In the case of discrete time system models, the stochastic case hardly amounts to more than introducing an extra sample point argument into each of the variables in the problem.

For continuous time models the situation is quite different. For deterministic continuous time modeling, ordinary (or partial) differential equations suffice as a modeling medium. It turns out that ordinary differential equations are inadequate for continuous time stochastic system models.

6.1 Introduction

For discrete time deterministic system models

\[ x_{k+1} = A x_k, \]

we can introduce randomness by simply adding a discrete time white noise to obtain

\[ x_{k+1} = A_k x_k + B_k u_k. \]

The “white noise” is an idealization of the notion of a “completely random” perturbation that disturbs the system trajectories.

If we want to do the same sort of thing for a continuous time model

\[ \frac{dx}{dt} = A(t) x(t), \]
we would like to add a “totally random, uncorrelated over time” disturbance to get something like
\[ \frac{dx}{dt} = A(t) x + B(t) u. \]
It turns out that this is doomed to failure, and that the “total randomness” notion is somehow fundamentally incompatible with the differentiability which is inherent in the solution of differential equations.

The solution for this problem turns out to be to abandon ordinary differential equations as the modeling tool, and to adopt what is known as stochastic differential equations as an alternative.

Stochastic differential equations are based on a continuous time stochastic process known as a Wiener process. This is an idealization of the physical process known as Brownian motion, describing the erratic motion of microscopic particles.

**DEFINITION** A Wiener process \( \{w(t)\} \) is a continuous time process meeting the following conditions:

- \( (w(t) - w(s)) \) for \( t > s \) is a mean zero, normal random variable, of variance
  \[ E((w(t) - w(s))^2) = (t - s), \]
- \( E(w(t)) = 0, \)
- \( w(0) = 0 \) (with probability 1),
- \( [w(t) - w(s)] \) is independent of \( [w(\tau) - w(\sigma)] \) provided that \( (t, s) \) is disjoint from \( (\tau, \sigma) \).

In view of the definition it is possible to write down an expression for the joint probability density of a sequence of successive samples of a Wiener process \( \{w(t_1), w(t_2), w(t_3), \ldots, w(t_n)\} \) for \( t_1 < t_2 < \ldots < t_n \) in the form
\[
p(w_1, w_2, \ldots w_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{((t_n - t_{n-1}) \ldots t_1)^{\frac{1}{2}}} e^{\frac{(w_n - w_{n-1})^2}{2(t_n - t_{n-1})}} e^{\frac{(w_{n-1} - w_{n-2})^2}{2(t_{n-1} - t_{n-2})}} \ldots e^{\frac{(w_2 - w_1)^2}{2(t_2 - t_1)}} e^{\frac{(w_1)^2}{2(t_1)}}.
\]

### 6.2 Stochastic Integrals

The basis for continuous time stochastic models is a stochastic integral. These take two forms, depending on whether the integrand is a deterministic function, or a