Wilson Bases on the Interval

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ABSTRACT In this chapter biorthogonal Wilson bases for $L^2([0, N])$ are investigated. The approach uses the even, periodic extension of functions defined on the interval. Starting from Wilson bases for periodic functions, Wilson bases for even, periodic functions are constructed. The basis functions are finally restricted to a suitable interval. Dual bases and Riesz bounds are given explicitly. The construction is based on a Zak transform for periodic functions and an unfolding operator for periodic Wilson bases. Fast algorithms for analysis and synthesis are described.

9.1 Introduction

Fourier methods have many applications in signal and image processing. The Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx, \quad \xi \in \mathbb{R},$$

of a function $f \in L^2(\mathbb{R})$ yields a frequency representation of $f$, i.e., $\hat{f}(\xi)$ represents the amplitude of the frequency term $e^{-2\pi i x \xi}$. The main disadvantage of the Fourier transform is that it does not represent local information in the time domain. A local perturbation of $f$ may result in a perturbation of $\hat{f}$ in the whole frequency domain.

To construct a transform with good localization in time and frequency, Gabor [13] proposed to represent a signal as a superposition of the elementary signals $e^{-(x-n)^2} e^{2\pi imx}$, $m, n \in \mathbb{Z}$. A more general approach is to consider the functions

$$g_{nm}(x) := g(x-an) e^{2\pi ibmx}, \quad m, n \in \mathbb{Z},$$

where $a$ and $b$ are some positive real numbers and $g \in L^2(\mathbb{R})$ is well localized around the origin of the time-frequency plane. To obtain a unique, numerical stable representation for each function $f \in L^2(\mathbb{R})$ by a linear combination of the $g_{nm}$, we need the Gabor system $\{g_{nm}\}$ to be a Riesz basis.

A sequence of functions $\{x_n : n \in \mathcal{I}\} \subset L^2(\mathbb{R})$, where $\mathcal{I}$ is a countable index set, is called a Riesz basis of $L^2(\mathbb{R})$ if it is complete and there exist
constants $A, B > 0$ such that, for every sequence of coefficients $\{a_n\} \in \ell^2(I)$,

$$A \sum_{n \in I} |a_n|^2 \leq \left\| \sum_{n \in I} a_n x_n \right\|_{L^2(\mathbb{R})}^2 \leq B \sum_{n \in I} |a_n|^2. \quad (9.1.1)$$

The constants $A$ and $B$ are called lower and upper Riesz bound, respectively. If $\{x_n\}$ is a Riesz basis, then for every $f \in L^2(\mathbb{R})$ there exists a unique expansion $f = \sum_{n \in I} a_n x_n$ which converges unconditionally in $L^2(\mathbb{R})$. This expansion is numerically stable. The ratio $\frac{B}{A}$ gives us a measure for the stability in the sense that one has good stability if $\frac{B}{A}$ is close to 1. The expansion coefficients $a_n$ can be determined by means of the dual basis $\{\tilde{x}_n : n \in I\}$, namely $a_n = \langle f, \tilde{x}_n \rangle$. The dual basis, which is also a Riesz basis, is uniquely determined by the biorthogonality condition $\langle x_n, \tilde{x}_m \rangle = \delta_{nm}$. In [19], Young gives several equivalent conditions for $\{x_n\}$ to be a Riesz basis. In particular, $\{x_n\}$ is a Riesz basis if and only if for every orthogonal basis $\{e_n : n \in I\}$ there exists a bounded invertible operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ so that $T e_n = x_n$. Furthermore, optimal Riesz bounds can be obtained from the norm of $T$ and $T^{-1}$. Namely, the inequalities in (9.1.1) are satisfied if and only if $0 < A \leq ||T^{-1}||^{-2}$ and $B \geq ||T||^2$ (cf. [4, Section 9.1]).

From the Balian–Low theorem [2, 9, 14] it follows that if the functions $g_{nm}$ constitute a Riesz basis of $L^2(\mathbb{R})$, which is possible only for $ab = 1$, then

$$\int_{\mathbb{R}} x^2 |g(x)| \, dx \int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)| \, d\xi = \infty,$$

i.e., $g$ has poor time-frequency localization. Only if one gives up uniqueness can one obtain Gabor frames with good time-frequency localization for $ab < 1$. For a detailed description of Gabor frames we refer to [12].

To construct orthonormal bases with good time-frequency localization properties, Wilson et al. [16, 18] suggested considering functions which are localized around the positive and negative frequency of the same order. This idea was used by Daubechies, Jaffard, and Journé [10] to construct orthonormal “Wilson bases” which consist of windowed sines and cosines (see section 9.2) instead of the exponentials $e^{2\pi ib \cdot}$. For such bases the disadvantage described in the Balian–Low theorem is completely removed. In [11], it is shown that these Wilson bases are unconditional bases for modulation spaces.

To obtain more freedom, Riesz bases are often investigated instead of orthonormal bases (which are Riesz bases with Riesz bounds $A = B = 1$). The additional freedom can be used to design bases with several desired features, e.g. good approximation properties in spaces of smooth functions. Such biorthogonal Wilson bases were studied by Coifman and Meyer in [8] for the particular case of a Gaussian window $w(x) := e^{-\zeta x^2}$. They proved that the Gaussian generates a Riesz basis if $\mathrm{Re} \, \zeta > 0$ and gave an explicit