Chapter 3

Relationships in a Triangle

1 Geometry of the triangle

We would like to develop some applications of the trigonometry we've learned to geometric situations involving a triangle.

Let us work with the three sides and three angles of the triangle.

\[ \triangle ABC \]

How many of these measurements do we need in order to reconstruct the triangle?!

This question is the subject of various “congruence theorems” in geometry. For example, if we know \( a, b \) and \( c \) (the three sides), the “SSS theorem” tells us that the three angles are determined. Any two triangles with the same three side-lengths are congruent.

But can we use any three side-lengths we like to make up our triangle? The “triangle inequality” of geometry tells us no. We must be sure that the

\[ A, B, \text{ and } C \]

are the vertices of a triangle, then we will also call the measures of the angles of the triangle \( A, B, \) and \( C \). Then the length of the sides opposite angles \( A, B, \) and \( C \) are called \( a, b, \) and \( c, \) respectively. There are also other “parts” of a triangle: its area, angle bisectors, altitudes, medians, and still more interesting lines and measurements.

\[ \text{[I. M. Gelfand et al., Trigonometry]} \]

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sum of any two of our three lengths is greater than the third; otherwise, the
sides don’t make a triangle. With this restriction, we can say that the three
side-lengths of a triangle determine the triangle.

What other sets of measurements can determine a triangle? A little
reflection will show that we will always need at least three parts (sides or
angles), and various theorems from geometry will help us in answering this
question.

Exercise

1. The table below gives several sets of data about a triangle. For
example, “ABa” means that we are discussing two angles and the
side opposite one of these angles. Some of the cases listed below are
actually duplicates of others.

<table>
<thead>
<tr>
<th>Data</th>
<th>Determine a triangle?</th>
<th>Restrictions?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>ABa</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Abb</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>ABc</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>AbC</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>ABC</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Abc</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Bbc</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Cbc</td>
<td></td>
</tr>
</tbody>
</table>

For each case, decide whether the given data determines a triangle.
What restrictions must we place on the data so that a triangle can be
formed? Some of these restrictions are a bit tricky. The case “abc”
was discussed above.

Please do not memorize this table! We just want you to recall what
geometry tells us – and what it does not tell us – about a triangle.

2 The congruence theorems and trigonometry

Some of the sets of data described above determine a triangle. For example,
“SAS” data (the lengths of two sides of a triangle and the measure of the