Chapter 6

Applications of Runge’s Theorem

1 The Mittag-Leffler theorem

The following exercise verifies a step in the proof of Theorem 1 in Section 1 of Chapter 6.

Exercise 279. With the notation as in the text, prove that for each open subset $\Omega \subseteq \mathbb{C}$ and for all compact subsets $K$ and $L$ of $\Omega$ such that

$$K \subset L^\circ \subset \Omega,$$

the following inclusion holds:

$$\hat{K}_\Omega \subset (L_\Omega)^\circ.$$

Exercise 280. Let $\{D_k : k \in I\}$ be an open cover of an open set $\Omega \subseteq \mathbb{C}$ by open discs. For each index $k \in I$, let $h_k \neq 0$ be a meromorphic function on $D_k$, and assume that for all indices $k, \ell \in I$, the function $g_{k, \ell} := h_k/h_\ell$ is holomorphic on $D_k \cap D_\ell$. Prove that for all indices $k, \ell \in I$ there exist holomorphic functions $f_k$ on $D_k$ without any zero and such that $f_k = g_{k, \ell} f_\ell$ on $D_k \cap D_\ell$.

Exercise 281. For each bounded connected open set $\Omega \subseteq \mathbb{C}$ and each bounded function $\varphi \in C^\infty(\Omega, \mathbb{C})$, not necessarily with compact support, prove that there exists a bounded function $u \in C^\infty(\Omega, \mathbb{C})$ such that $\partial u/\partial \overline{z} = \varphi$ on $\Omega$.

Exercise 282. Let $R_1$ and $R_2$ be rectangles in $\mathbb{C}$ whose union $R_1 \cup R_2$ is also a rectangle. Suppose that $f : (R_1 \cap R_2) \rightarrow \mathbb{C}$ is a bounded holomorphic function on their intersection. Prove that there exist bounded holomorphic functions $f_1 \in \mathcal{H}(R_1)$ and $f_2 \in \mathcal{H}(R_2)$ such that $f = f_1 - f_2$ on $R_1 \cap R_2$. 

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2 The cohomology form of Cauchy's theorem

Exercise 283. Provide examples of domains $\Omega \subseteq \mathbb{C}$ with the following properties.

(283.1) The cokernel of the derivative map $d : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ has prescribed dimension $k \in \mathbb{N}$.

(283.2) The cokernel of the derivative map $d : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ has infinite dimension.

3 The theorem of Weierstrass

Exercise 284. Consider a simply connected open subset $\Omega \subseteq \mathbb{C}$ and a meromorphic function $f$ on $\Omega$ with the set of poles of $f$ denoted by $E$. Assume that the residue of $f$ at each point of $E$ is an integer.

Prove that there exists a meromorphic function $g$ on $\Omega$ that is holomorphic on $\Omega \setminus E$ and such that the function

$$\frac{g'}{g} - f \in \mathcal{H}(\Omega \setminus E)$$

has a primitive in $\mathcal{H}(\Omega \setminus E)$. Deduce that if all the poles of $f$ are simple, then there exists a meromorphic function $h$ on $\Omega$ such that on $\Omega \setminus E$

$$\frac{h'}{h} = f.$$

The following standard exercises relate "infinite" products (limits of finite products) to "infinite" series (limits of finite sums).

Exercise 285.

(285.1) Prove that for each sequence of complex numbers $(w_\ell)_{\ell=0}^\infty$, the product

$$\prod_{\ell=1}^\infty (1 + |w_\ell|) := \lim_{L \to \infty} \prod_{\ell=1}^L (1 + |w_\ell|) = P$$

converges to a limit $P$ if but only if the series

$$\sum_{\ell=1}^\infty |w_\ell| := \lim_{L \to \infty} \sum_{\ell=1}^L |w_\ell| = S$$

converges to a limit $S$. 