§13. The principles of symplectic mechanics

Nonrelativistic symplectic mechanics

(13.0) Consider a classical dynamical system. An elementary description of such a system consists of decomposing it into material points and formulating the laws describing the forces to which these points are subjected. If these laws satisfy Maxwell's principle (12.90), the procedures which we have studied in the preceding section (namely, the construction of the evolution space and its Lagrange form, the integration of the equations of motion, and the passage to the quotient) lead to the construction of a symplectic manifold: the space of motions.

We will now give up the notions of material point and force and will take as axioms of mechanics some essential properties of the space of motions:222

I. The space of motions of a dynamical system is a connected symplectic manifold.

II. If several dynamical systems evolve independently, the manifold of motions of the composite system is the symplectic direct product of the spaces of motions of the component systems.

III. If a dynamical system is isolated, its manifold of motions admits the Galilei group as a dynamical group.

As one can see, we not only give up the notion of a material point but also the notion of an evolution space (and thus a fortiori the notions of phase space, "configuration space", etc.).223 Euclidean space and time itself are only indirectly invoked through the Galilei group.

Since the principles (13.1) are valid in classical mechanics (see (12.100), (12.146), and (12.117), respectively), they do not break with classical mechanics, but simply enlarge it. They permit us to treat new dynamical systems, as we shall see in examples later on.

Of course it is the analogy with the classical cases which will guide us in our physical interpretation of these new systems.

222Definitions: symplectic manifold (9.1), direct product of symplectic manifolds (9.7), Galilei group (12.73), and dynamical group (11.1).

223Giving up these notions is familiar in quantum mechanics.
(13.2) Consider an isolated dynamical system. We shall suppose that its structure is completely characterized by its symplectic properties and by the manner in which the Galilei group \( G \) acts on it. In more precise terms, two isolated dynamical systems having spaces of motions \( U \) and \( U' \) will be called isomorphic if there exists a symplectomorphism \( \Phi \) from \( U \) to \( U' \) such that
\[
\Phi(a_U(x)) = a_{U'}(\Phi(x)) \quad \forall x \in U, \forall a \in G.
\]
When this is the case, we will say that the dynamical systems have the same model and that \( \Phi \) is an isomorphism from the first system to the second.

Moments, mass, and the center of mass

The study of Galilean moments of an isolated system made in the preceding section remains valid as long as the explicit expression of the Lagrange form (12.45) does not play any role in it. Thus the Galilei group \( G \) of matrices
\[
a \equiv \begin{bmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \quad A \in \text{SO}(3), \quad b \in \mathbb{R}^3, \quad c \in \mathbb{R}^3, \quad e \in \mathbb{R}
\]
acts on the space of motions \( U \) of an isolated dynamical system while preserving the Lagrange form (13.1.III). To each element of the Lie algebra \( \mathcal{G} \) of \( G \)
\[
Z \equiv \begin{bmatrix} j(\omega) & \beta & \omega \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix} \quad \omega, \beta, \gamma \in \mathbb{R}^3, \quad \varepsilon \in \mathbb{R}
\]
there corresponds a vector field \( Z_U \) on \( U \) which is an infinitesimal canonical transformation. The associated moment (definition (11.7)) has been denoted by
\[
\mu \equiv \{l, g, p, E\},
\]
and its action on the Lie algebra \( \mathcal{G} \) is given by
\[
\mu(Z) \equiv \{l, \omega\} - \{g, \beta\} + \{p, \gamma\} - E \varepsilon.
\]
(13.7) \[ We now make the fundamental assumption that this moment exists.\]224 We thus know that the dynamical group \( G \) possesses a symplectic cohomology class (§11), which can be labeled by a number \( m \) (12.132). On

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224One might say that this is a new principle (see (12.143)). Newtonian mechanics can borrow this principle from relativistic mechanics without being alienated from it (see (13.81)).