Chapter 9: Extension of the Schauder estimates

We describe here, from the point of view of the paradifferential calculus as used in Chapters 3 and 8, a refinement of the Schauder estimates due to Nirenberg. We also describe an important application of this to Monge-Ampere equations, given in [CNS].

§9.1. Nirenberg’s refinement

In this section we will consider solutions to nonlinear elliptic equations on a compact manifold $M$, without boundary. The regularity result of Theorem 2.2.G for solutions to

$$ F(x, D^m u) = f, \tag{9.1.1} $$

assumed to be elliptic, implies the estimate

$$ \|u\|_{C^{m+r}} \leq C(\|u\|_{C^{m+s}}) \left[ \|f\|_{C^s} + \|u\|_{C^s} + 1 \right] \tag{9.1.2} $$
given $\varepsilon > 0$, $s < m + r$. Similarly, given $1 < p < \infty$,

$$ \|u\|_{H^{m+r,p}} \leq C(\|u\|_{C^{m+s}}) \left[ \|f\|_{H^{s,p}} + \|u\|_{H^{s,p}} + 1 \right]. \tag{9.1.3} $$

Nirenberg [N2] showed how to replace $C(\|u\|_{C^{m+s}})$ by a factor depending on any modulus of continuity of $D^m u$, possibly somewhat weaker than any Hölder estimate on $D^m u$. We will give a proof of such an estimate here, via a modification of the analysis used to establish Theorem 3.3.C.

Given a continuous monotonic function $\varphi : [0, 1] \to \mathbb{R}$, $\varphi(0) = 0$, which we will call a modulus, we will say $u \in C_\varphi$ if $u$ is bounded and

$$ |u(x) - u(y)| \leq C\varphi(|x - y|) \text{ for } |x - y| \leq 1. \tag{9.1.4} $$

We use the norm

$$ \|u\|_{C_\varphi} = \|u\|_{L^\infty} + \sup_{|x-y| \leq 1} |u(x) - u(y)| / \varphi(|x - y|). \tag{9.1.5} $$

Given $m \in \mathbb{Z}^+$, we say $u \in C_\varphi^m$ if $D^\alpha u \in C_\varphi$ for $|\alpha| \leq m$. We can use the norm

$$ \|u\|_{C_\varphi^m} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{C_\varphi}. \tag{9.1.6} $$

Recall the formula

$$ F(x, D^m u) = M(u; x, D) + F(x, D^m \Psi_0(D)u) \tag{9.1.7} $$
with

\[ (9.1.8) \]

\[ M(u; x, \xi) = \sum_{|\alpha| \leq m} \sum_j \int_0^1 (\partial F/\partial \zeta_\alpha)(\Psi_j D^m u + \tau \psi_{j+1} u) d\tau \cdot \psi_{j+1}(\xi) \xi^\alpha. \]

As shown in §3.3,

\[ (9.1.9) \]

\[ u \in C^m \implies M(u; x, \xi) \in A^0 S^m_{1,1}. \]

Recall the symbol smoothing of §1.3:

\[ (9.1.10) \]

\[ M(u; x, \xi) = M^b(x, \xi) + M^s(x, \xi), \quad M^s(x, \xi) \in S^m_{1,\delta}. \]

We choose \( \delta \in (0,1) \). Thus, for \( p \in (1, \infty) \), \( s \in \mathbb{R} \),

\[ (9.1.11) \]

\[ ||M^s(x, D)v||_{H^{s,p}} \leq C sp(||u||_{C^m})||v||_{H^{s+m,p}}. \]

Furthermore, a parametrix \( P(x, D) \in OPS^{1,\delta}_{1,\delta} \) for \( M^s(x, D) \), as in (3.3.26), satisfies an estimate

\[ (9.1.12) \]

\[ ||P(x, D)v||_{H^{s+m,p}} \leq C sp(||u||_{C^m})||v||_{H^{s,p}}. \]

Also \( P(x, D)M^s(x, D) = I + S \) where \( S \) satisfies the same sort of estimate. Now (9.1.3) results from the estimate

\[ (9.1.13) \]

\[ ||M^b(x, D)v||_{H^{s,p}} \leq C sp(||u||_{C^m + \varepsilon})||v||_{H^{s+m-\varepsilon,p}} \]

valid for

\[ 0 < \delta < \varepsilon, \ s > 0. \]

It is this estimate we need to modify for \( u \in C^m_{\varphi} \).

**Lemma 9.1.A.** If \( u \in C^m_{\varphi} \), \( 1 < p < \infty \), \( s > 0 \), then there exist \( K(\varepsilon) = K_{\varphi}(\varepsilon) \), such that

\[ (9.1.14) \]

\[ ||M^b(x, D)v||_{H^{s,p}} \leq C sp(||u||_{C^m_{\varphi}}) \cdot [\varepsilon||v||_{H^{s+m,p}} + K(\varepsilon)||v||_{L^p}]. \]

for \( 0 < \varepsilon \leq 1. \)

**Proof:** Given \( \eta \in (0,1] \), write \( M^b(x, \xi) = A_\eta(x, \xi) + B_\eta(x, \xi) \), with \( A_\eta(x, \xi) = M^b(x, \xi) \Phi(\eta \xi) \), given \( \Phi \in C^\infty_0(\mathbb{R}^n) \), \( \Phi(\xi) = 1 \) for \( |\xi| \leq 1, 0 \) for \( |\xi| \geq 2 \). Since \( M^b(x, \xi) \in S^m_{1,1} \), we hence have a bound on \( \eta^{m+s} A_\eta(x, \xi) \) in \( S^m_{1,1} \) for \( \eta \in (0,1] \), and consequently an estimate

\[ (9.1.15) \]

\[ ||A_\eta(x, D)v||_{H^{s,p}} \leq C sp(||u||_{C^m}) \eta^{-m-s}||v||_{L^p}. \]