Chapter 1: Symbols with limited smoothness

Here we establish some very general facts about symbols $p(x, \xi)$ with limited smoothness in $x$. We prove some operator bounds on $p(x, D)$ when $p(x, \xi)$ is homogeneous in $\xi$.

In §1.2 we show that these simple results lead to some easy regularity theorems, for solutions to an elliptic PDE, $F(x, D^m u) = f$, under the assumption that $u$ already possesses considerable smoothness, e.g., roughly $2m$ derivatives. Though this is a rather weak result, which will be vastly improved in Chapters 2 and 3, nevertheless it has some uses, beyond providing a preliminary example of techniques to be developed here. For example, when examining local solvability of $F(x, D^m u) = f$, one can use a Banach space implicit function theorem to find $u \in H^s$ with $s$ large, and then apply such a regularity result as Theorem 1.2.D to obtain local $C^\infty$ solutions.

One key tool for further use of symbols introduced here is to write $p(x, \xi) = p^\#(x, \xi) + p^\flat(x, \xi)$, with $p^\#(x, \xi)$ a $C^\infty$ symbol, in $S^m_{1,5}$, and $p^\flat(x, \xi)$ having lower order. This symbol decomposition is studied in §1.3.

§1.1. Symbol classes

We introduce here some general classes of symbols $p(x, \xi)$ which have limited regularity in $x$. To start with, let $X$ be any Banach space of functions, such that

$$C_0^\infty \subset X \subset C^0.$$  

We say

$$p(x, \xi) \in X S^m_{1,0} \iff \|D_\xi^\alpha p(\cdot, \xi)\|_X \leq C_\alpha \langle \xi \rangle^{m-|\alpha|}, \quad \alpha \geq 0.$$  

For applications, we will generally want $X$ to be a Banach algebra under pointwise multiplication, and more specifically

$$f \in C^\infty(\mathbb{R}), \quad u \in X \implies f(u) \in X;$$

$$f \text{ maps bounded sets to bounded sets in } X.$$  

Such an $X$ as we consider will usually be one of a family $\{ X^s : s \in \Sigma \}$ of spaces, known as a scale. The set $\Sigma$ will be of the form $[\sigma_0, \infty)$ or $(\sigma_0, \infty)$, and we assume

$$X^s \subset X^t \text{ if } t < s,$$

provided $t, s \in \Sigma$, and, if $s \in \Sigma, m \in \mathbb{Z}^+$, then $s + m \in \Sigma$ and

$$OPD^m : X^{s+m} \rightarrow X^s,$$
where \( OPD^m \) denotes the space of differential operators of order \( m \) (with smooth coefficients).

Examples of scales satisfying the conditions above are

\[
X^s = C^s(\mathbb{R}^n), \quad \Sigma = [0, \infty)
\]

and

\[
X^s = H^{s,p}(\mathbb{R}^n), \quad \Sigma = (n/p, \infty),
\]

for any given \( p \in [1, \infty) \). In (1.1.6), \( C^s \) is the space of \( C^k \) functions whose \( k^{th} \) derivatives satisfy the Hölder condition

\[
|u(x + y) - u(x)| \leq C|y|^\sigma, \quad |y| \leq 1,
\]

where \( s = k + \sigma, \ 0 \leq \sigma < 1 \). The spaces (1.1.7) are Sobolev spaces.

We say \( \{X^s\} \) is microlocalizable if, for \( m \in \mathbb{R}, \ s, s + m \in \Sigma \),

\[
OPS_{1,0}^m : X^{s+m} \longrightarrow X^s.
\]

The Sobolev spaces (1.1.7) have this property provided \( p \in (1, \infty) \). The property (1.1.9) fails for the spaces \( C^s \) if \( s \) is an integer. In such a case, one needs to use the Zygmund spaces

\[
X^s = C^s_*(\mathbb{R}^n), \quad \Sigma = (0, \infty),
\]

which coincide with \( C^s \) is \( s \) is not an integer, but differ from \( C^s \) if \( s \) is an integer. Some important properties of Sobolev spaces and Zygmund spaces are discussed in Appendix A.

We will say \( p(x, \xi) \in XS^m, \) or merely \( XS^m, \) provided \( p(x, \xi) \in XS^m_{1,0} \) and \( p(x, \xi) \) has a classical expansion

\[
p(x, \xi) \sim \sum_{j \geq 0} p_j(x, \xi)
\]

in terms homogeneous of degree \( m - j \) in \( \xi \) (for \( |\xi| \geq 1 \)), in the sense that the difference between \( p(x, \xi) \) and the sum over \( j < N \) belongs to \( XS^{m-N}_{1,0} \).

As usual, we define the operator associated to \( p(x, \xi) \)

\[
p(x, D)u = \int p(x, \xi) \hat{u}(\xi) e^{ix \cdot \xi} d\xi.
\]

We also consider \( p(D, x) \), defined by

\[
p(D, x)u = (2\pi)^{-n} \int \int p(y, \xi) e^{i(x-y) \cdot \xi} u(y) \, dy \, d\xi.
\]