The Feynman diagram expansion of relativistic field theories gives rise to integrals that diverge at short distances or equivalently at large momenta. To have a well-defined theory we need to regulate these divergences. There are many ways to do this, but one that has both practical and conceptual advantages is to put the theory on a discrete space-time lattice. In this chapter we will explore this.

15.1 Boson Fields

Put a real variable \( \phi(n) \) at each point \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \) on a \( d \)-dimensional square lattice whose points are a distance \( a \) apart.
We consider the integral

$$Z = \int d[\varphi] \exp \left\{ -\sum_{n \in \mathbb{Z}^d} \sum_{i=1}^d a_i^d \left[ \frac{1}{2} \frac{(\varphi(n + i) - \varphi(n))^2}{a^2} + \frac{1}{2} m^2 \varphi^2(n) \right] \right\}$$  \hspace{1cm} (15.1)

In this expression $d[\varphi]$ is shorthand for $\prod_n d\varphi(n)$ and $i$ is the vector $(\ldots, 1, \ldots)$ with the 1 in the $i$th place. One should think of $Z$ as the partition function of some statistical system of springs and masses. The exponent is clearly intended to be an approximation to

$$L = \int d^d x \left\{ \frac{1}{2} (\partial \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right\}.$$  \hspace{1cm} (15.2)

We are really interested in the Green functions

$$\langle \hat{\varphi}(n_1) \ldots \hat{\varphi}(n_n) \rangle = \frac{1}{Z} \int d[\varphi] \{ \varphi(n_1) \ldots \varphi(n_n) \} e^{\left\{ -\sum_n \sum_{i=1}^d (\ldots) \right\}}.$$  \hspace{1cm} (15.3)

We can evaluate these by the simple but rather unilluminating procedure of Fourier transforming $\varphi(n)$. Let us temporarily set $a = 1$, so that it does not clutter our expressions. We can always restore it later by dimensional analysis. We now write $\varphi(n)$ in terms of its Fourier components:

$$\varphi(n) = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \tilde{\varphi}(k) e^{in \cdot k}.$$  \hspace{1cm} (15.4)

Because of the discrete cubic lattice the momenta $k$ are restricted to lie in the Brillouin zone.