2.1 Introduction

"I see it, but I don’t believe it!” This disbelief of Georg Cantor in his own creations exemplifies the great skepticism that his work on infinite sets inspired in the mathematical community of the late nineteenth century. With his discoveries he single-handedly set in motion a tremendous mathematical earthquake that shook the whole discipline to its core, enriched it immeasurably, and transformed it forever. Besides disbelief, Cantor encountered fierce opposition among a considerable number of his peers, who rejected his discoveries about infinite sets on philosophical as well as mathematical grounds.

Beginning with Aristotle (384–322 B.C.E.), two thousand years of Western doctrine had decreed that actually existing collections of infinitely many objects of any kind were not to be part of our reasoning in philosophy and mathematics, since they would lead directly into a quagmire of logical contradictions and absurd conclusions. Aristotle’s thinking on the infinite was in part inspired by the paradoxes of Zeno of Elea during the fifth century B.C.E. The most famous of these asserts that Achilles, the fastest runner in ancient Greece, would be unable to surpass a much slower runner, provided that the slower runner got a bit of a head start. Namely, Achilles would then first have to cover the distance between the starting positions, during which time the slower runner would advance a certain distance. Then Achilles would have to cover that distance, while the slower runner would again advance, and so on. Even though the distances would be getting very small, there are infinitely many of them, so that it would take Achilles infinitely long to cover all of them [4, p. 179]. Aristotle deals with this and Zeno’s other paradoxes (Exercise 2.1) at great length, concluding that the way to resolve them is to deny the possibility of collecting infinitely many objects into a complete and actually existing whole. The only allowable concept is the so-called potential in-
finite. While it is inadmissible to consider the complete collection of all natural numbers, it is allowed to consider a finite collection of such numbers that can be enlarged as much as one wishes. As an analogy, when we study a function $f(x)$ of a real variable $x$, then we may be interested in the behavior of $f(x)$ as $x$ becomes arbitrarily large. But we are not allowed simply to replace $x$ by $\infty$ to find out, because we might get a nonsensical result.

A paradox of a more overtly mathematical nature, given by Galileo Galilei (1564–1642) in the seventeenth century, shows that there are just as many perfect squares as there are natural numbers, by pairing off each number with its square:

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 4 & 9 & 16
\end{array}
$$

But on the other hand, since not all natural numbers are perfect squares, there are clearly more natural numbers than perfect squares. Galileo even observes that the perfect squares become ever sparser as one progresses through the natural numbers, making the pairing above even more paradoxical. He concludes that "the attributes 'larger,' 'smaller,' and 'equal' have no place either in comparing infinite quantities with each other or comparing infinite with finite quantities" [65, p. 33].

A comprehensive account of the philosophical struggle with the concept of the infinite from early Greek thought to the present can be found in [123] (see also [145]).

It was left to a theologian from Prague in the early nineteenth century to make a systematic study of mathematical paradoxes involving infinity. After studying philosophy, physics, and mathematics at the University of Prague, Bernard Bolzano (1781–1848) decided that his calling was to be a theologian, even though he had been offered a chair in mathematics. As a professor of theology at the University of Prague, beginning in 1805, Bolzano nonetheless spent part of his time pursuing mathematical research. After being dismissed from his position for expressing allegedly heretical opinions in his sermons, and being prohibited from ever teaching or publishing again, he used his enforced leisure to work almost exclusively on mathematics. His philosophical interests had always drawn him to questions about the foundations of mathematics, its definitions, methods of proof, and the nature of its concepts. Thus, he naturally was led to the philosophy of the infinite. Not only did he conclude that mathematics was well equipped to deal with infinite sets in a systematic manner, free of contradictions, he even went as far as arguing that mathematics was the proper realm in which to discuss and resolve all paradoxes involving the infinite.\(^1\)

\(^1\)In India, on the other hand, the Jaina mathematics of the middle of the first millenium B.C.E. not only entertained the idea that many different kinds of infinity could exist, but actually developed the beginnings of a system of infinite sets and infinite numbers [91, pp. 18, 218–219, 249–253].