If $E \rightarrow X$ is a vector bundle, then it is of considerable interest to investigate the special operation derived from the functor "multilinear alternating forms." Applying it to the tangent bundle, we call the sections of our new bundle differential forms. One can define formally certain relations between functions, vector fields, and differential forms which lie at the foundations of differential and Riemannian geometry. We shall give the basic system surrounding such forms. In order to have at least one application, we discuss the fundamental 2-form, and in the next chapter connect it with Riemannian metrics in order to construct canonically the spray associated with such a metric.

We assume throughout that our manifolds are Hausdorff, and sufficiently differentiable so that all of our statements make sense.

V, §1. VECTOR FIELDS, DIFFERENTIAL OPERATORS, BRACKETS

Let $X$ be a manifold of class $C^p$ and $\varphi$ a function defined on an open set $U$, that is a morphism

$$\varphi: U \rightarrow \mathbb{R}.$$  

Let $\xi$ be a vector field of class $C^{p-1}$. Recall that

$$T_x\varphi: T_x(U) \rightarrow T_x(\mathbb{R}) = \mathbb{R}$$

is a continuous linear map. With it, we shall define a new function to be denoted by $\xi \varphi$ or $\xi \cdot \varphi$, or $\xi(\varphi)$. (There will be no confusion with this notation and composition of mappings.)

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Proposition 1.1. There exists a unique function $\xi \varphi$ on $U$ of class $C^{p-1}$ such that

$$(\xi \varphi)(x) = (T_x \varphi)\xi(x).$$

If $U$ is open in the Banach space $E$ and $\xi$ denotes the local representation of the vector field on $U$, then

$$(\xi \varphi)(x) = \varphi'(x)\xi(x).$$

Proof. The first formula certainly defines a mapping of $U$ into $\mathbb{R}$. The local formula defines a $C^{p-1}$-morphism on $U$. It follows at once from the definitions that the first formula expresses invariantly in terms of the tangent bundle the same mapping as the second. Thus it allows us to define $\xi \varphi$ as a morphism globally, as desired.

Let $F_u^p$ denote the ring of functions (of class $C^p$). Then our operation $\varphi \mapsto \xi \varphi$ gives rise to a linear map

$$\partial_{\xi}: F_u^p(U) \to F_u^{p-1}(U),$$

defined by $\partial_{\xi} \varphi = \xi \varphi$.

A mapping $\partial: R \to S$ from a ring $R$ into an $R$-algebra $S$ is called a derivation if it satisfies the usual formalism: Linearity, and $\partial(ab) = a\partial(b) + \partial(a)b$.

Proposition 1.2. Let $X$ be a manifold and $U$ open in $X$. Let $\xi$ be a vector field over $X$. If $\xi(x) \neq 0$, then $\xi(x) = 0$ for all $x \in U$. Each $\xi$ is a derivation of $F_u^p(U)$ into $F_u^{p-1}(U)$.

Proof. Suppose $\xi(x) \neq 0$ for some $x$. We work with the local representations, and take $\varphi$ to be a continuous linear map of $E$ into $\mathbb{R}$ such that $\varphi(\xi(x)) \neq 0$, by Hahn–Banach. Then $\varphi'(y) = \varphi$ for all $y \in U$, and we see that $\varphi'(x)\xi(x) \neq 0$, thus proving the first assertion. The second is obvious from the local formula.

From Proposition 1.2 we deduce that if two vector fields induce the same differential operator on the functions, then they are equal.

Given two vector fields $\xi, \eta$ on $X$, we shall now define a new vector field $[\xi, \eta]$, called their bracket product.

Proposition 1.3. Let $\xi, \eta$ be two vector fields of class $C^{p-1}$ on $X$. Then there exists a unique vector field $[\xi, \eta]$ of class $C^{p-2}$ such that for each open set $U$ and function $\varphi$ on $U$ we have

$$[\xi, \eta] \varphi = \xi(\eta(\varphi)) - \eta(\xi(\varphi)).$$