Starting from Maxwell’s equations, we want to treat electromagnetic fields that are rapidly alternating in time and space. At first, we will restrict ourselves to fields in vacuum. In this case, \( \mathbf{D} = \mathbf{E} \) and \( \mathbf{B} = \mathbf{H} \); furthermore, \( j = 0 \) and \( \rho = 0 \). So, we have Maxwell’s equations in the form

\[
\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \tag{15.1}
\]

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \tag{15.2}
\]

\[
\nabla \cdot \mathbf{E} = 0 \tag{15.3}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{15.4}
\]

We may eliminate the vectors \( \mathbf{E} \) and \( \mathbf{B} \) from the system of equations. Taking the curl of the equation (15.1):

\[
\nabla \times (\nabla \times \mathbf{B}) = \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{E})
\]

and substituting the equation (15.2), then

\[
\nabla \times (\nabla \times \mathbf{B}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{15.5}
\]

Using the identity \( \nabla \times (\nabla \times \mathbf{B}) = \nabla (\nabla \cdot \mathbf{B}) - \Delta \mathbf{B} \) due to \( \nabla \cdot \mathbf{B} = 0 \) we obtain from equation (15.5)

\[
\Delta \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{15.6a}
\]

Correspondingly, we may eliminate \( \mathbf{B} \) and obtain

\[
\Delta \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{15.6b}
\]
Hence, both vectors \( \mathbf{E} \) and \( \mathbf{B} \) fulfill the same wave equation. We have encountered wave equations already in mechanics in treating oscillation processes. There are various types of solutions for these equations. At first, we notice that the form of the wave equation

\[
\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(\mathbf{r}, t) = 0
\]

is relativistically invariant because the operator

\[
\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \sum_{\mu=1}^{4} \frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial x_{\mu}}
\]

is equal to the scalar product of the four-gradient

\[
\frac{\partial}{\partial x_{\mu}} = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial ct} \right\}
\]

with itself. An equation of the form (15.7) is solved by any function

\[
u(\mathbf{r}, t) = u(k_{\mu} x_{\mu}) = u(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)
\]

if \( k_{\mu} k_{\mu} \equiv \sum_{\mu} k_{\mu}^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 = 0 \), that is, \( k_{\mu} \) is a four-null-vector (light vector). We verify easily that

\[
\frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial x_{\mu}} u(k_{\nu} x_{\nu}) = k_{\mu} k_{\mu} \frac{d^2 u(z)}{dz^2}, \quad z = k_{\nu} x_{\nu}
\]

and therefore,

\[
0 = \left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(k_{\nu} x_{\nu}) = \frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial x_{\mu}} u(k_{\nu} x_{\nu}) = k_{\mu} k_{\mu} \frac{d^2 u(z)}{dz^2}
\]

Writing \( \mathbf{k} = \{ k_{\mu} \} = \{ k_1, k_2, k_3, i \omega / c \} = \{ \mathbf{k}, i \omega / c \}, \) the dispersion relation is \( k_{\mu} k_{\mu} = 0 = k^2 - \frac{\omega^2}{c^2} \) or \( k^2 = \frac{\omega^2}{c^2}, \) and thus \( k = \omega / c. \) Then, the solutions of the wave equations are: \( u(z) = u(k_{\mu} x_{\mu}) = u(\mathbf{k} \cdot \mathbf{r} - \omega t). \) This may be understood in the framework of a one-dimensional wave equation,

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x \pm ct) = 0
\]

Here, \( u(x \pm ct) \) are again arbitrary functions possessing always the special combination

\[
x + ct = \{ x, i ct \} \cdot \{ 1, - i \} = \{ x, i ct \} \cdot \{ k_1', k_4' \}
\]

\[\overset{\text{1}}{\text{We use Einstein’s sum convention. This means to sum up automatically over identical indices } k_{\mu} k_{\mu} = \sum_{\mu=1}^{4} k_{\mu}^2.}\]