In this chapter we will study the solution of systems of nonlinear equations. As opposed to linear equations, no explicit solution techniques are, in general, available for nonlinear equations, and hence their solution completely relies on iterative methods. In the first section we shall begin with the application of the Banach fixed point theorem for systems of nonlinear equations with one or several variables. Given the fact that iterative techniques have a long history in mathematics, the significance of Banach’s fixed point theorem originates from its unified approach, covering a wide variety of different successive approximation methods.

In the second section, we will continue with the study of Newton’s iteration method for finding zeros of functions of one or several variables. This iteration scheme is attributed to Newton, since in 1669 he developed a solution method for cubic equations by linearization that may be viewed as a precursor of what is now known as Newton iteration. He also used this method for approximately solving Kepler’s equations for planetary motion.

In the concluding two sections of this chapter we will consider the application of Newton’s method for finding zeros of polynomials and its modification into the more recently developed Levenberg–Marquardt scheme for solving the least squares problem.

Given the vast number of iterative methods available for nonlinear equations, we will confine our presentation to describing the fundamental ideas and will not aim at a complete treatment of the subject.
6.1 Successive Approximations

In this section, we will consider systems of \( n \) nonlinear equations for \( n \) unknowns of the form

\[
f(x) = x,
\]

where \( x = (x_1, \ldots, x_n)^T \) and \( f(x) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)^T \).

We begin by studying the case of a single nonlinear equation with one unknown. Obviously, in one dimension, solving \( f(x) = x \) geometrically corresponds to determining the intersection of the graph of the function \( f \) with the straight line described by the function \( x \mapsto x \).

**Theorem 6.1** Let \( D \subset \mathbb{R} \) be a closed interval and let \( f : D \to D \) be a continuously differentiable function with the property

\[
q := \sup_{x \in D} |f'(x)| < 1.
\]

Then the equation \( f(x) = x \) has a unique solution \( x \in D \), and the successive approximations

\[
x_{\nu+1} := f(x_\nu), \quad \nu = 0, 1, 2, \ldots,
\]

with arbitrary \( x_0 \in D \) converge to this solution. We have the a priori error estimate

\[
|x_\nu - x| \leq \frac{q^\nu}{1 - q} |x_1 - x_0|
\]

and the a posteriori error estimate

\[
|x_\nu - x| \leq \frac{q}{1 - q} |x_\nu - x_{\nu-1}|
\]

for all \( \nu \in \mathbb{N} \).

**Proof.** Equipped with the norm \( \| \cdot \| = | \cdot | \) the space \( \mathbb{R} \) is complete. By the mean value theorem, for \( x, y \in D \) with \( x < y \), we have that

\[
f(x) - f(y) = f'((\xi))(x - y)
\]

for some intermediate point \( \xi \in (x, y) \). Hence

\[
|f(x) - f(y)| \leq \sup_{\xi \in D} |f'(\xi)||x - y| = q|x - y|,
\]

which is also valid for \( x, y \in D \) with \( x \geq y \). Therefore, \( f \) is a contraction, and the assertion follows from the Banach fixed point Theorem 3.46.

Figure 6.1 illustrates graphically the successive approximations for functions \( f \) with positive and negative slope, respectively, of absolute value less than one. Note that the sequence \( (x_\nu) \) converges to the fixed point...