Chapter 2

Finite Min-Max and Constrained Optimization

We devote this chapter to optimality conditions and algorithms for solving three classes of progressively more difficult optimization problems: min-max problems of the form

$$\min \max f^j(x), \quad x \in \mathbb{R}^n, j \in \mathcal{Q}$$

(0a)

inequality constrained optimization problems of the form

$$\min \{f^0(x) \mid f^j(x) \leq 0, \ j \in \mathcal{Q}\}$$

(0b)

and inequality and equality constrained optimization problems of the form

$$\min \{f^0(x) \mid f^j(x) \leq 0, \ j \in \mathcal{Q}, g^k(x) = 0, \ k \in \mathcal{R}\}$$

(0c)

where the constraint functions $f^j : \mathbb{R}^n \to \mathbb{R}, j \in \mathcal{Q}$, and $g^k : \mathbb{R}^n \to \mathbb{R}, k \in \mathcal{R}$ are continuously differentiable, while the cost function $f^0 : \mathbb{R}^n \to \mathbb{R}$ can be either continuously differentiable on $\mathbb{R}^n$ or a max function of the form

$$f^0(x) = \max_{k \in \mathcal{P}} c^k(x),$$

(0d)

with the $c^k : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable.

In passing, we note that the problem MMP is equivalent to the problem

$$\min \{x^{n+1} \mid f^j(x) - x^{n+1} \leq 0, \ j \in \mathcal{Q}\}$$

(0e)

which is of the form ICP. Later, we will also see that, given a problem of the form ICP, one can construct a problem of the form MMP that is locally equivalent to it. These observations indicate that min-max problems are not only important in their own right, but that they also provide a natural bridge from unconstrained optimization problems to constrained optimization.
problems. Hence we will deal with them first. In fact, we will obtain optimality conditions and some algorithms for constrained optimization problems directly from those for min-max problems.

As in Section 1.1, we will again present necessary optimality conditions in three equivalent forms. The first form is the most basic. It expresses the fact that to first or second-order, the cost must increase at feasible points sufficiently close to a local minimizer. The second form is a consequence of the first form. For first-order conditions, it consists of an equation involving gradients with coefficients that are called multipliers. In the case of second-order conditions, a quadratic inequality involving the multipliers and second derivatives is added to the first-order conditions. The easiest way to verify whether optimality conditions are satisfied in either the first or second form is to set up an auxiliary optimization problem with quadratic cost and affine inequality and equality constraints. The value of this auxiliary optimization problem is a function of \( x \), the point at which the verification is being carried out. We will call this value function an optimality function. We will see that optimality functions, as defined by us, are nonpositive-valued and are zero only at points \( x \) satisfying the first (and hence also the second) form of optimality conditions. Thus, the third form of optimality conditions is in the form of zeros of an optimality function. Our favorite optimality functions are based on a strictly convex, first-order local model for an optimization problem and have three important advantages: (i) they are continuous, (ii) their evaluation yields a continuous cost descent direction, and (iii) their value can be used to compute upper and lower bounds on the minimum value that is being sought.

Our presentation will make constant use of the mathematical material in the Sections 5.1 - 5.5, and the reader is therefore advised to read these sections before proceeding any further.

### 2.1 Optimality Conditions for Min-Max

We begin by developing first- and second-order optimality conditions for the min-max problem \( \text{MMP} \) which can be rewritten as

\[
\min_{x \in \mathbb{R}^n} \psi(x) , \quad (1a)
\]

with

\[
\psi(x) \triangleq \max_{j \in q} f^j(x) \quad (1b)
\]

and the functions \( f^j : \mathbb{R}^n \to \mathbb{R} \) continuously differentiable. We recall that necessary conditions must be satisfied by any local minimizer and that a point satisfying necessary conditions is called a stationary point. Sufficient conditions imply that a point is a local minimizer. Unlike the situation in the unconstrained