Each problem considered previously reduces to that of optimizing (usually minimizing) a real valued function $J$ defined on a subset $\mathcal{D}$ of a linear space $\mathcal{V}$. In the present chapter we shall view problems in this context and introduce the associated directional derivatives (Gâteaux variations) of the functions which will be required for what follows. We begin with a catalogue of standard linear spaces presupposing some familiarity with vector space operations, with continuity, and with differentiability in $\mathbb{R}^d$.

§2.1. Real Linear Spaces

All functions considered in this text are assumed to be real valued or real vector valued. The principal requirement of a real linear (or vector) space of functions is that it contain the sums and (real) scalar multiples of those functions. We remark without proof that the collection of real valued functions $f$, $g$, on a (nonempty) set $S$ forms a real linear space (or vector space) with respect to the operations of pointwise addition:

$$(f + g)(x) = f(x) + g(x), \quad \forall \ x \in S$$

and scalar multiplication:

$$(cf)(x) = cf(x), \quad \forall \ x \in S, \quad c \in \mathbb{R}. \ \ ([N]).$$

Similarly, for each $d = 1, 2, \ldots$ the collection of all $d$-dimensional real vector valued functions on this set $S$ forms a linear space with respect to the following operations of componentwise addition and scalar multiplication: if $F = (f_1, f_2, \ldots, f_d)$ and $G = (g_1, \ldots, g_d)$, where $f_j$ and $g_j$ are real valued functions

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on $S$ for $j = 1, 2, \ldots, d$, so that $F(x) = (f_1(x), f_2(x), \ldots, f_d(x))$, and $G(x) = (g_1(x), g_2(x), \ldots, g_d(x)), \forall x \in S$, then

$$(F + G)(x) = F(x) + G(x)$$

and

$$(cF)(x) = cF(x) = (cf_1(x), cf_2(x), \ldots, cf_d(x)), \quad \forall x \in S.$$ 

It follows that each subspace of these spaces, i.e., each subset which is closed under the defining operations of addition and scalar multiplication, is itself a real linear space.

In particular, if continuity is definable on $S$, then $C(S) (= C^0(S))$, the set of continuous real valued functions on $S$, will be a real linear space since the sum of continuous functions, or the multiple of a continuous function by a real constant, is again a continuous function. Similarly, for each open subset $D$ of Euclidean space and each $m = 1, 2, \ldots, C^m(D)$, the set of functions on $D$ having continuous partial derivatives of order $\leq m$, is a real linear space, since the laws of differentiation guarantee that the sum or scalar multiple of such functions will be another. In addition, if $D$ is bounded with boundary $\partial D$, and $\overline{D} = D \cup \partial D$, then $C^m(\overline{D})$, the subset of $C^m(D) \cap C(\overline{D})$ consisting of those functions whose partial derivatives of order $\leq m$ each admit continuous extension to $\overline{D}$, is a real linear space.

For example, when $a, b \in \mathbb{R}$, then $[a, b] = [a, b]$, is a closed and bounded interval. A function $y$, which is continuous on $[a, b]$, is in $C^1[a, b]$ if it is continuously differentiable in $(a, b)$ and its derivative $y'$ has finite limiting values from the right at $a$ (denoted $y'(a^+)$) and from the left at $b$ (denoted $y'(b^-)$). When no confusion can arise we shall use the simpler notations $y'(a)$ and $y'(b)$, respectively, for these values, with a similar convention for higher derivatives at $a, b$, when present. Observe that $y_0(x) = x^{3/2}$ does define a function in $C^1[0, 1]$ while $y_1(x) = x^{1/2}$ does not.

Finally, for $d = 1, 2, \ldots, [C(S)]^d$, $[C^m(D)]^d$, and $[C^m(\overline{D})]^d$, the sets of $d$-dimensional vector valued functions whose components are in $C(S)$, $C^m(D)$, and $C^m(\overline{D})$, respectively, also form real linear spaces.

We know that subsets $\mathcal{D}$ of these spaces provide natural domains for optimization of the real valued functions in Chapter 1. However, these subsets do not in general constitute linear spaces themselves. For example,

$$\mathcal{D} = \{ y \in C[a, b] : y(a) = 0, y(b) = 1 \}$$

is not a linear space since if $y \in \mathcal{D}$ then $2y \notin \mathcal{D}$. ($2y(b) = 2(1) = 2 \neq 1.$) However,

$$\mathcal{D}_0 = \{ y \in C[a, b] : y(a) = y(b) = 0 \}$$

is a linear space. (Why?)

In the sequel we shall assume the presence of a real linear space $\mathcal{Y}$ consist-

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1 We abbreviate $C((a, b))$ by $C(a, b)$, $C^1([a, b])$ by $C^1[a, b]$, etc.