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Are There Functions Defining Prime Numbers?

To obtain prime numbers, it is natural to ask for functions $f$ defined for all natural numbers $n \geq 1$, which are computable in practice and produce some or all prime numbers.

For example, one of the following conditions should be satisfied:

(a) $f(n) = p_n$; (the $n$th prime) for all $n \geq 1$.
(b) $f(n)$ is always a prime number, and if $n \neq m$, then $f(n) \neq f(m)$.
(c) The set of prime numbers is equal to the set of positive values assumed by the function.

Clearly, condition (a) is more demanding than (b) and (c).

The results reached have been rather disappointing, except for some of theoretical importance related to condition (c).

Another idea is to consider functions whose values are easily computable, like polynomials with integral coefficients, and to study when they have many prime values. As will be seen, it is a rather fruitful idea.

I. Functions Satisfying Condition (a)

In their famous book, Hardy and Wright asked:

(1) Is there a formula for the $n$th prime number?
(2) Is there a formula for a prime, in terms of the preceding prime?

What is intended in (1) is to find a closed expression for the $n$th prime $p_n$, in terms of $n$, by means of functions that are computable and, if possible, classical. Intimately related with
these problems is to find reasonable expressions for the function counting primes.

For every real number \( x > 0 \) let \( \pi(x) \) denote the number of primes \( p \) such that \( p \leq x \).

This is a traditional notation for one of the most important functions in the theory of prime numbers. I shall return to it in Chapter 4. Even though the number \( \pi = 3.14 \ldots \) and the function \( \pi(x) \) do occur below in the same formula, this does not lead to any ambiguity.

Despite the nil practical value of the formulas which will be presented, I tend to believe that such formulas may have some relevance to logicians who wish to understand clearly how various parts of arithmetic may be deduced from different axiomatizations or from fragments of Peano’s arithmetic.

**Formulas for \( \pi(n) \)**

In 1964, Willans gave the following formula for the prime counting function. It is based on the classical Wilson theorem which I proved in Chapter 2.

For every integer \( j \geq 1 \) let

\[
F(j) = \left[ \cos^2 \pi \frac{(j - 1)! + 1}{j} \right].
\]

So for any integer \( j > 1 \), \( F(j) = 1 \) when \( j \) is a prime, while \( F(j) = 0 \) otherwise. Also \( F(1) = 1 \).

Thus,

\[
\pi(n) = -1 + \sum_{j=1}^{n} F(j).
\]

Willans also expressed

\[
\pi(n) = \sum_{j=2}^{n} H(j) \quad \text{for } n = 2, 3, \ldots,
\]

where

\[
H(j) = \frac{\sin^2 \pi \frac{(j - 1)!}{j}^2 / j}{\sin^2(\pi/j)}.
\]

Mináč gave an alternate (unpublished) expression, which in-