Completeness and compactness

In classical analysis the Cauchy criterion plays a critically important role as a test for convergence. In the theory of abstract metric spaces its role is no less important, but here it serves as a basis for classifying metric spaces.

Definition. An infinite sequence \( \{x_n\} \) (indexed by either \( \mathbb{N} \) or \( \mathbb{N}_0 \)) in a metric space \((X, \rho)\) is a Cauchy sequence, or satisfies the Cauchy criterion, if \( \lim_{m,n} \rho(x_m, x_n) = 0 \), i.e., if for any positive number \( \varepsilon \) there exists an index \( N \) such that \( \rho(x_m, x_n) < \varepsilon \) for all \( m, n \geq N \). (Equivalently, if \( T_n \) denotes the tail \( \{x_k : k \geq n\} \) of \( \{x_n\} \), then \( \{x_n\} \) is Cauchy if and only if \( \lim_n \text{diam } T_n = 0 \).

The basic facts about Cauchy sequences are quickly established.

Proposition 8.1. Let \((X, \rho)\) be a metric space. Every convergent sequence in \( X \) is Cauchy, and every Cauchy sequence in \( X \) is bounded. Furthermore, every Cauchy sequence in \( X \) that possesses a convergent subsequence is itself convergent to the limit of that subsequence. Consequently, a Cauchy sequence can have at most one cluster point, and if it possesses a cluster point, it must converge to that cluster point.

Proof. Suppose first that \( \{x_n\} \) is a convergent sequence in \( X \) and that \( \lim_n x_n = a_0 \). If \( \varepsilon > 0 \) is given, then there exists an index \( n_0 \) such that \( \rho(x_n, a_0) < \varepsilon / 2 \) for all \( n \geq n_0 \), whence it follows by the triangle inequality that \( \rho(x_m, x_n) < \varepsilon \) for all \( m, n \geq n_0 \). Thus every convergent sequence

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is Cauchy. On the other hand, if \( \{x_n\} \) is Cauchy, then there exists an index \( N \) such that \( \rho(x_m, x_n) < 1 \) for all \( m, n \geq N \). But then, if we set \( K = \sup_{n < N} \rho(x_n, x_N) \), the entire sequence \( \{x_n\} \) is contained in the ball \( D_{K+1}(x_N) \), and is therefore bounded. To complete the proof, it suffices to show that a Cauchy sequence converges to the limit of any of its convergent subsequences (see Proposition 6.5). Suppose then that \( \{x_n\} \) is Cauchy and \( \{y_k = x_{n_k}\} \) is a subsequence of \( \{x_n\} \) that converges to a point \( a_1 \). If \( \varepsilon > 0 \) is given, then there exists an index \( N \) such that \( \rho(x_m, x_n) < \varepsilon/2 \) whenever \( m, n \geq N \), and an index \( k_0 \) such that \( \rho(y_k, a_1) < \varepsilon/2 \) for all \( k \geq k_0 \). But then, if \( k_1 \) is any index such that \( k_1 \geq k_0 \) and \( n_{k_1} \geq N(k_1 = N \lor k_0, \text{for example}) \), we have

\[
\rho(x_n, a_1) \leq \rho(x_n, y_{k_1}) + \rho(y_{k_1}, a_1) < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

for all \( n \geq N \).

\( \square \)

**Definition.** A metric space \( X \) is complete if every Cauchy sequence in \( X \) is convergent to some point of \( X \).

It should be acknowledged at once that completeness is not a rare or exceptional property. Indeed, all of the metric spaces of classical analysis are complete.

**Example A.** The metric space \( \mathbb{R} \) is complete. If \( \{t_n\} \) is a Cauchy sequence in \( \mathbb{R} \), then \( \{t_n\} \) is bounded and therefore possesses a convergent subsequence (Ex. 6L). But then \( \{t_n\} \) is itself convergent. The metric space \( \mathbb{C} \) is also complete. (If \( \{\alpha_n\} \) is a Cauchy sequence in \( \mathbb{C} \), then both of the sequences \( \{\text{Re } \alpha_n\} \) and \( \{\text{Im } \alpha_n\} \) are Cauchy in \( \mathbb{R} \) and therefore convergent in \( \mathbb{R} \). But then \( \{\alpha_n\} \) is convergent in \( \mathbb{C} \).) More generally, for much the same reasons, the spaces \( \mathbb{R}^n \) and \( \mathbb{C}^n \) of Examples 6A and 6B are complete. A normed space that is complete as a metric space (Prob. 6A) is a Banach space. Thus \( \mathbb{R}^n \) and \( \mathbb{C}^n \) are, respectively, real and complex Banach spaces.

**Example B.** If \( Z \) is a set and \((X, \rho)\) is a complete metric space, then the space \( B(Z; X) \) of all bounded mappings of \( Z \) into \( X \) is complete in the metric \( \rho_\infty \) of uniform convergence (Ex. 6H). Indeed, if \( \{\phi_n\} \) is a sequence in \( B(Z; X) \) that is Cauchy with respect to \( \rho_\infty \), then the sequence \( \{\phi_n(z)\} \) is Cauchy, and therefore convergent, in \( X \) for each \( z \) in \( Z \). Let us write \( \phi(z) = \lim_n \phi_n(z) \) for all \( z \) in \( Z \). We shall show that \( \phi \in B(Z; X) \) and that \( \lim_n \rho_\infty(\phi, \phi_n) = 0 \), thus proving the assertion. To this end let \( \varepsilon \) be an arbitrarily prescribed positive number, and let \( N \) be an index such that \( \rho_\infty(\phi_m, \phi_n) < \varepsilon \) whenever \( m, n \geq N \). Then for an arbitrary element \( z \) of \( Z \) we have \( \rho(\phi_m(z), \phi_n(z)) < \varepsilon \) for all \( m, n \geq N \) and, letting \( n \) tend to infinity, we conclude that

\[
\rho(\phi_m(z), \phi(z)) \leq \varepsilon, \quad m \geq N,
\]