Principle of Virtual Work

A structure loaded by external forces, like surface tractions and body forces, is considered to be in a state of equilibrium. The free-body diagram for such a set of conditions is given in Fig. 5.1. Necessarily, the resulting density of the force in every material point must vanish, $f = 0$. A material point with position vector $r$ is given a virtual displacement $\delta r$, where $|\delta r| \ll l_{\text{char}}$ (characteristic length of the body), and the (elementary) virtual work per unit of volume vanishes consequently

$$f \cdot \delta r = 0.$$

If we assume the virtual displacements are compatible with the integrity of the continuum, integration over the material volume causes the total virtual work of the internal and external forces to vanish.

Fig. 5.1. Virtual displacements. Variation of the equilibrium configuration. $0$ is a point of reference.
where \( \int_V k \cdot \delta r \, dV \) is the virtual work of the (given) external body forces.

The second volume integral in Eq. (5.1) is changed by means of the *Gauss* integral theorem (the symmetry of the stress tensor is understood, and \( \delta r = \delta u \) )

\[
\sum_i \int_V \left( \frac{\partial \sigma_i}{\partial x_i} \cdot \delta r \right) dV = \sum_i \int_V \frac{\partial}{\partial x_i} (\sigma_i \cdot \delta r) \, dV - \sum_i \int_V \sigma_i \cdot \frac{\partial}{\partial x_i} (\delta r) \, dV
\]

\[
= \int_{\partial V} (\sigma_n \cdot \delta r) \, dS - \frac{1}{2} \sum_i \sum_j \int_V \sigma_{ij} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dV .
\]

The surface integral

\[
\int_{\partial V} (\sigma_n \cdot \delta r) \, dS
\]

is the virtual work done by the external surface tractions. Since the virtual displacements are assumed to be sufficiently small, the linearized geometric relations of Eq. (1.21) apply to the virtual variations of the strains \( \delta \varepsilon_{ij} \),

\[
\delta W^{(0)} = - \sum_i \sum_j \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV .
\]

Formally, a comparison with the differential increment of the elastic potential Eq. (3.36) in an elastic body is illustrative.

Hence, if we denote the virtual work of the external forces (body forces and surface tractions) by \( \delta W^{(e)} \), the vanishing sum of the virtual work of the internal and external forces is rewritten in shorthand notation

\[
\delta W = \delta W^{(0)} + \delta W^{(e)} = 0 .
\]

Equation (5.3) remains valid if on some parts of the body surface, the displacements are prescribed and, hence, are not varied virtually, \( \delta r = 0 \) at these parts of the surface, ie the surface tractions (supporting forces) that are necessary to produce the prescribed displacements do not contribute to the virtual work of the external forces.

If Eq. (5.3) holds for all possible and admissible virtual displacement fields measured from a reference configuration, the principle of virtual work (in the special version, it is called the