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The Idea of Independence, with Applications

To him, therefore, the succession to the Norland estate was not so really important as to his sisters; for their fortune, independent of what might arise to them from their father's inheriting that property, could be but small.

Jane Austen, Sense and Sensibility

5.1 Independence of events

In the real world, we frequently encounter pairs of events such that the occurrence of one, we feel, has no bearing or influence on the occurrence of the other. For example, suppose I toss a coin once and note what comes up. Suppose then I repeat the procedure, giving the coin another toss and again note what comes up. Consider the events

\[ H_1 = \{\text{head on toss 1}\}, \quad H_2 = \{\text{head on toss 2}\}. \]

Under most circumstances most people would have the strong intuitive feeling that the occurrence of \( H_1 \) gives you no information about whether or not \( H_2 \) will occur. The same can be said about the second roll of a pair of dice, say, where rolling a 7 (or any other value) does not appear to affect what will happen when we roll again. In these cases, we describe our feelings by saying that the second outcome is independent of the first outcome. We can include this notion of independence in our mathematical model in the following way: since the conditional probability of \( H_2 \) given
the intuitive notion of independence suggests that if the occurrence of $H_1$ has no effect on the occurrence of $H_2$, this conditional probability should be the same as the unconditional probability. In symbols,

$$P(H_2/H_1) = P(H_2).$$  \hspace{1cm} (5.1)$$

Put formula 5.1 into the left-hand side of the conditional probability formula above and multiply both sides of the resulting equation by $P(H_1)$ to obtain the famous \textit{product rule} for independent events

$$P(H_2) \cdot P(H_1) = P(H_2 \cap H_1).$$

Events that do not satisfy the product rule, or equivalently, formula 5.1, are called \textit{dependent}. This notion of independence introduced into our mathematical model turns out to be very fruitful. Most of the classical theory of probability was done under assumptions of independence; it is only relatively recently in the subject that various forms of dependence conditions have been studied extensively.

Let's note an interesting symmetry arising from the mathematics. We have said that $H_2$ is independent of $H_1$ because $H_1$ happened first and intuition demands that a first event may or may not affect the occurrence of a second event, not the other way around. Nothing, however, prevents us from considering $P(H_1/H_2)$, the conditional probability of a head on toss 1 given that a head on toss 2 occurred. Evaluate this using the conditional probability formula while at the same time assuming $H_2$ is independent of $H_1$ to get

$$P(H_1/H_2) = \frac{P(H_1 \cap H_2)}{P(H_2)} = \frac{P(H_1) \cdot P(H_2)}{P(H_2)} = P(H_1),$$

which is to say that our assumption of $H_2$ independent of $H_1$ implies that $H_1$ is independent of $H_2$, so the idea of independence is symmetric: as soon as a first event is known to be independent of a second, the second is automatically independent of the first. Of course, independence or dependence in our model simply means that a conditional probability is equal to an unconditional one or is not, and is not required to make intuitive sense in the real-life application of the model: what does it mean, you may ask, for a first toss to be influenced or not by a second toss? As mathematicians, we really don't have to worry about this question. As philosophers or physicists, we may find this interesting to speculate upon. The mathematics does not distinguish between the forward and backward directions of time. Because of this symmetry, we can simply say of two events that they are independent, without having to specify which of the two events is the conditioning one.