In gambling, one thing you should never do is take something for granted.

John Scarne, Scarne's Guide to Casino Gambling

6.1 The problem of points

In this chapter, we discuss several games; more along this line will follow in the next chapter. The first problem, solved by both Pascal and Fermat, goes back to the earliest days of probability as a formal theory. Suppose two people are playing a game with the winner receiving prize money at the end. If the game is forced to end before either player wins, how should the prize money be divided between the players? Pascal introduced the principle that the prize money should be divided in proportion to each player’s conditional probability of winning if the game were to be continued, given the score when the game is forced to end. Suppose, for example, that the plays of the game constitute a sequence of Bernoulli trials where $A$ wins a point with probability $p$ (success) and $B$ wins a point with probability $1 - p$ (failure), and $n$ points are needed to win. We will not derive the general formula but will give the solution for the case where $A$ has $n - 1$ points and $B$ has $n - 2$ points. Then $A$ needs one point to win and $B$ needs two points. $A$ can win in two ways if the game were to be continued at this moment: (1) $A$ can win the next point, and (2) $B$ can win the next point and $A$ can win the succeeding point. This gives the value $p + p(1 - p)$ for the conditional probability of $A$ winning. If $p = 1 - p = .5$ and the purse is $100$, then,
according to Pascal's principle, $A$ should receive $75$ and $B$ $25$.

### 6.2 Craps

Craps is played with a pair of dice. The player (sometimes called the “shooter”) rolls once. If the dice show 7 or 11, she wins. If the dice show 2, 3, or 12, she loses. If the dice show any other value, this number is known as the gambler's “point.” She must now keep rolling the dice until either she gets 7 before her point appears, in which case she loses, or else gets her point before 7 appears, in which case she wins. In addition to the shooter, most real-life craps games have a host of other people betting on the shooter's game.

We are going to calculate the probability of the event “the gambler wins at craps.” This is an interesting game to analyze because the sample space is rather complicated. A typical element of the sample space can be considered an $n$-tuple $(x_1, x_2, \ldots, x_n)$ of $n$ rolls of the dice, where the entry $x_i$ denotes the number appearing on roll $i$, and the game ends at roll $n$. Using this notation, the simplest elements of the sample space can be written: $(7)$, $(11)$, $(2)$, $(3)$, $(12)$. Suppose the first roll is 4; this becomes the gambler's point. The sample space contains elements of the form $(4, x_2, \ldots, x_n)$ where the term $x_n$ is either 7 or 4 and the terms $x_2$ through $x_{n-1}$ must be different from both 7 and 4. The totality of such elements can be described as the event “the first roll is 4, and the game ends at roll $n$.” Let us calculate the probability of the event “the first roll is 4, and the gambler wins at roll $n$.” For this to happen, there must have been a sample point of the type $(4, x_2, \ldots, x_{n-1}, 4)$, where the final 4 is in the $n$th position, and the $n - 2$ terms between the two 4's may not be either 7 or 4. A roll of 4 on two dice can occur in three ways out of 36 possible rolls, so the probability of the initial 4 as well as the terminal 4 is $3/36$. The probability of each of the $n - 2$ terms between the 4's is $27/36$, since nine of the 36 rolls are excluded (six ways for 7, three ways for 4). By independence of the rolls, the product rule gives us

$$P(\text{the first roll is 4, and the gambler wins at roll } n) = \left( \frac{3}{36} \right)^2 \cdot \left( \frac{27}{36} \right)^{n-2},$$

valid for $n \geq 2$.

Now suppose we want to find the probability of the event “the first roll is 4, and the gambler wins at some time.” This event is the disjoint union of the events that the gambler wins at time $n$ for the times $n = 2, 3, \ldots$. We are already experts at this sort of thing [if you don't agree with this, go back and look at formulas 5.3 and 5.4 of Chapter 5], so we know

$$P(\text{the first roll is 4, and the gambler wins at some time}) \quad (6.1)$$