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The Earliest Number Theory

The Sumerians and Babylonians

The Sumerians lived in the southern part of Mesopotamia (Iraq). About 2000 B.C., their civilisation was absorbed by the Babylonians, and Babylonian culture reached its peak about 575 B.C., under Nebuchadnezzar. The mathematical achievements we shall discuss in this chapter are recorded on the clay tablets of the Sumerians and Babylonians. Most of these achievements go back as far as 2000 B.C. — about the time when Abraham’s father was living in the Sumerian city of Ur. We shall use the word ‘Babylonian’ for what is perhaps more accurately described as ‘Mesopotamian’ mathematics.

The Babylonians used a counting scale, not of 10, but of 60, and this scale was taken over into Greek astronomy by Hipparchus of Nicaea (about 150 B.C.). It is thanks to the Babylonians, and Hipparchus, that we have 60 minutes in an hour. According to the prophet Ezekiel (573 B.C.), in the ancient system of weights, scale 60 was endorsed by God himself:

Lord Yahweh says this: ... Twenty shekels, twenty-five shekels and fifteen shekels are to make one mina (Ezekiel 45:9-12).

The Babylonians could solve linear and quadratic equations. They could even solve the simultaneous equations

\[ x^8 + x^6 y^2 = (3,200,000)^2 \]
\[ xy = 1,200 \]
The Babylonians built pyramid-shaped ‘ziggurats’. The first story of a ziggurat might measure \( n \times n \times 1 \), the second story \((n - 1) \times (n - 1) \times 1 \), and so on — with the top two stories measuring \( 2 \times 2 \times 1 \) and \( 1 \times 1 \times 1 \). The volume of such a ziggurat is

\[
1^2 + 2^2 + \cdots + (n - 1)^2 + n^2
\]

and the Babylonians knew that this equals

\[
\frac{n(n + 1)(2n + 1)}{6}
\]

a result first proved by Archimedes (287-212 B.C.).

The Bible tells us that there was once an attempt to build a ziggurat ‘with its top reaching heaven’ (Genesis 11:4). Perhaps the promoters of the Tower of Babel mistakenly believed that the infinite series \( 1^2 + 2^2 + 3^2 + \cdots \) converges.

The Babylonians knew the formulas for the areas of the triangle, trapezium, and circle. According to a clay tablet found in Susa in 1936, they used the value \( 3\frac{1}{8} \) for \( \pi \).

**Pythagorean Triples**

A triple \((x, y, z)\) of positive integers, with \( x, y < z \) gives the lengths of the sides of a right angled triangle if and only if \( x^2 + y^2 = z^2 \). Although such triples are called *Pythagorean triples*, they were studied by the Babylonians, long before Pythagoras (525 B.C.). From a clay tablet called *Plimpton 322*, we know that the Babylonians were interested in a certain kind of Pythagorean triple, which we shall call a *Babylonian triple*. The triple \((x, y, z)\) is a *Babylonian triple* just in case the lengths \( x, y, \) and \( z \) can be expressed in the form

\[
2uv, \quad u^2 - v^2, \quad \text{and} \quad u^2 + v^2
\]

with \( u \) and \( v \) relatively prime positive integers having no prime factors other than 2, 3, and 5 (the prime divisors of the Babylonian scale 60). The numbers \( u \) and \( v \) are *generating numbers*. As the Babylonians realised,

\[
(2uv)^2 + (u^2 - v^2)^2 = (u^2 + v^2)^2
\]

and hence the coordinates of a Babylonian triple are the lengths of the sides of a right-angled triangle, a *Babylonian triangle*.

For example, \((56, 90, 106)\) is a Babylonian triple (with \( u = 9 \) and \( v = 5 \)), but \((28, 45, 53)\) is not (since we would have \( u = 7 \) with \( u \) having a prime factor other than 2, 3, and 5).