22
The Later Middle Ages

Fibonacci and the Rabbits

In the thousand years from 300 to 1300, there was only one outstanding European mathematician, namely, Leonardo of Pisa (1180–1250), who was known as Fibonacci. He learned his mathematics in Algeria, where his father was a custom-house officer.

In 1202, Fibonacci published his *Liber Abaci* in which he explained the Arabic system of numerals we now use and gave the Rabbit Problem:

Suppose that rabbit pregnancy lasts one month, and that every female rabbit gets pregnant at the beginning of every month, from the time she is one month old on. Suppose that female rabbits always give birth to two bunnies, one male and one female. How many pairs of rabbits will you have on January 2, 1203 if you start with a newborn pair on January 1, 1202?

The number of rabbit pairs increases as follows:

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\]

If \(F_n\) is the \(n\)th *Fibonacci number*, we have

\[F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}\]
De Moivre’s Formula

The following theorem was discovered by Abraham de Moivre, about 1730.

**Theorem** Let \( s = \frac{1+\sqrt{5}}{2} \) — this is the ‘golden ratio’. Then the \( n \)th Fibonacci number is the natural number nearest \( s^n/\sqrt{5} \).

**Proof:** Let \( r = \frac{1-\sqrt{5}}{2} \). Then \( r + s = 1 \) and \( rs = -1 \). Thus

\[
F_{n+1} - rF_n = s(F_n - rF_{n-1}) \\
= s^2(F_{n-1} - rF_{n-2}) \\
= \ldots \\
= s^{n-1}(F_2 - F_1) \\
= s^{n-1}(1 - r) \\
= s^n
\]

Similarly, \( F_{n+1} - sF_n = r^n \). Subtracting, we obtain \( (s - r)F_n = s^n - r^n \), so that

\[
F_n = s^n/\sqrt{5} - r^n/\sqrt{5}
\]

Furthermore, \( r^n/\sqrt{5} \) is close to 0, and it is not hard to show now that \( F_n \) is the natural number closest to \( s^n/\sqrt{5} \).

For example, the 10th Fibonacci number is the integer nearest 55.0036, in other words, 55.

**The Liber Quadratorum**

In 1225, Emperor Frederick II organised a mathematics contest. Leonardo answered all the questions correctly, winning easily. Two of the problems were the following:

1. solve \( x^3 + 2x^2 + 10x = 20 \)
2. find a rational \( a/b \) such that \((a/b)^2 \pm 5\) are both squares of rationals.

If \( k \) is a positive integer such that, for some rational \( a/b \) both \((a/b)^2 \pm k\) are squares of rationals, then \( k \) is congruent. Note that if \( k \) and \( x \) are positive integers, then \( k \) is congruent if and only if \( kx^2 \) is congruent. Problem (2) of the contest was the problem of showing that 5 is congruent.

In solving Problem 19 of Book III of the *Arithmetica*, Diophantus notes that in a right triangle,

\[
(\frac{1}{2} \text{ hypotenuse})^2 \pm \text{ area} = \text{ a square}
\]