CHAPTER XIV

Inverse Mappings and Differential Equations

XIV, §1. THE INVERSE MAPPING THEOREM

Both the inverse mapping theorem and the existence theorem for differential equations will be based on a basic and simple lemma in complete metric spaces.

Lemma 1.1 (Shrinking Lemma). Let $M$ be a complete metric space, and let $T: M \to M$ be a mapping. Assume that there exists a number $K$ with $0 < K < 1$ such that for all $x, y \in M$ we have

$$d(Tx, Ty) \leq Kd(x, y),$$

where $d$ is the distance function in $M$. Then $T$ has a unique fixed point $z$, that is a point such that $Tz = z$. If $x \in M$, then

$$z = \lim_{n \to \infty} T^n x.$$

Proof. For simplicity of notation, we assume that $M$ is a closed subset of a Banach space. We first observe that a fixed point $z$, if it exists, is unique because if $z_1$ is also fixed, then

$$|z - z_1| = |Tz - Tz_1| \leq K|z - z_1|,$$

so $z - z_1 = 0$. Now for existence, let $m, n$ be positive integers and say $n \geq m, n = m + r$. Then for any $x$ we have

$$|T^n x - T^m x| \leq K^m |x - T^r x|$$
and

$$|x - T'x| \leq |x - Tx| + |Tx - T^2x| + \cdots + |T^{r-1}x - T'r|x|$$

$$\leq (1 + K + \cdots + K^{r-1})|x - Tx|.$$

This shows that the sequence \( \{T^n x\} \) is Cauchy, converging to some element \( z \in M \). This element \( z \) is a fixed point because

$$|Tz - TT^n x| \leq K|z - T^n x|$$

and for \( n \) sufficiently large, \( TT^n x \) approaches \( z \) and also \( Tz \). This proves the shrinking lemma.

We shall call \( K \) in the lemma a shrinking constant for \( T \).

Let \( U \) be open in a Banach space \( E \), and let \( f: U \rightarrow F \) be a \( C^p \) map \((p \geq 1)\). We shall say that \( f \) is a \( C^p \)-isomorphism or is \( C^p \)-invertible on \( U \) if the image \( f(U) \) is an open set \( V \) in \( F \), and if there exists a \( C^p \) map

$$g: V \rightarrow U$$

such that \( g \circ f \) and \( f \circ g \) are the identity maps on \( U \) and \( V \) respectively. We say that \( f \) is a local \( C^p \)-isomorphism at a point \( x \) in \( U \), or is locally \( C^p \)-invertible at \( x \), if there exists an open set \( U_1 \) contained in \( U \) and containing \( x \) such that the restriction of \( f \) to \( U_1 \) is \( C^p \)-invertible on \( U_1 \).

It is clear that the composite of two \( C^p \)-isomorphisms is again a \( C^p \)-isomorphism, and that the composite of two locally \( C^p \)-invertible maps is also locally \( C^p \)-invertible. In other words, if \( f \) is locally \( C^p \)-invertible at \( x \), if \( f(x) \) is contained in some open set \( V \), and if \( g: V \rightarrow G \) is locally \( C^p \)-invertible at \( f(x) \), then \( g \circ f \) is locally \( C^p \)-invertible at \( x \).

The inverse mapping theorem provides a criterion for a map to be locally \( C^p \)-invertible, in terms of its derivative.

**Theorem 1.2 (Inverse Mapping Theorem).** Let \( U \) be open in a Banach space \( E \), and let \( f: U \rightarrow F \) be a \( C^p \) map. Let \( x_0 \in U \) and assume that \( f'(x_0): E \rightarrow F \) is a toplinear isomorphism (i.e. invertible as a continuous linear map). Then \( f \) is a local \( C^p \)-isomorphism at \( x_0 \).

**Proof.** Let \( \lambda = f'(x_0) \). It suffices to prove that \( \lambda^{-1} \circ f \) is locally invertible at \( x_0 \) because we may consider \( \lambda^{-1} \circ f \) instead of \( f \) itself. Thus we have reduced our theorem to the case where \( E = F \) and \( f'(x_0) \) is the identity mapping. Next, making translations, it suffices to prove our theorem when \( x_0 = 0 \) and \( f(x_0) = 0 \) also. From now on, we make these additional assumptions.

Let \( g(x) = x - f(x) \). Then \( g'(0) = 0 \) and by continuity there exists