After the abstract theory on arbitrary measured spaces, it is a relief to get into some classical situations on $\mathbb{R}^n$ where we see the integral at work. None of this chapter will be used later, except for the approximation by Dirac families in the uniqueness proof for the spectral measure of Chapter 20.

In this chapter we deal with the Fourier transform in a context of absolute convergence. In Chapter 10, §2 we shall deal with a more delicate context, involving oscillatory convergence.

**VIII, §1. CONVOLUTION**

Suppose first we deal with functions $f$, $g$ on the real line. We shall study their **convolution**, defined by the integral

$$f \ast g(y) = \int_{-\infty}^{\infty} f(x)g(y - x) \, dx.$$  

Of course, the integral must be convergent, or even absolutely convergent. Theorems 1.1 and 1.2 will give conditions for such convergence. Furthermore, we don’t need to work only on $\mathbb{R}$, and we shall express the results on $\mathbb{R}^n$, abbreviating

$$\int_{\mathbb{R}^n} f(x)g(y - x) \, dx = \int f(x)g(y - x) \, dx.$$  

In most applications, one of the two functions $f$ or $g$ is continuous or
even $C^\infty$, and the resulting convolution is also continuous or $C^\infty$. To see this one must be able to take a limit or differentiate under the integral sign, and the next section gives basic conditions under which this is legitimate. We shall see several examples after the main approximation theorem is proved in Theorem 3.1.

We now come to the basic tests for absolute convergence of the convolution integral.

**Theorem 1.1.** Let $f, g \in L^1(\mathbb{R}^n)$. Then for almost all $y \in \mathbb{R}^n$ the function

$$x \mapsto f(x)g(y - x)$$

is in $L^1(\mathbb{R}^n)$. The convolution $f * g$ given for almost all $y$ by

$$f * g(y) = \int f(x)g(y - x) \, dx$$

is also in $L^1$. The association $(f, g) \mapsto f * g$ is an associative, commutative bilinear map, satisfying

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Thus $L^1(\mathbb{R}^n)$ is a Banach algebra under the convolution product.

**Proof.** We integrate $|f(x)||g(y - x)|$ first with respect to $y$, and then with respect to $x$. We apply part 2 of Fubini's theorem, Theorem 8.7 of Chapter VI. We then conclude that $f * g$ is in $L^1$. The last inequality in the statement of the theorem follows at once. The bilinearity is obvious, and so is commutativity. The associativity is proved using Fubini's theorem, and is left to the reader.

**Theorem 1.2.** Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then $f * g(y)$ is defined by the integral for almost all $y$ and is in $L^p$. We have

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

**Proof.** The case $p = 1$ is treated in Theorem 1.1. Suppose that $p = \infty$. If $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, so we may assume $g$ is a bounded measurable function, then we may also form the convolution $f * g$ given by the same formula

$$f * g(y) = \int f(x)g(y - x) \, dx = \int f(y - x)g(x) \, dx.$$

The integrals converge absolutely, and we have the trivial estimate from