Chapter 1

Introduction to topology

“when George drew out a tin of pineapple from the bottom of the hamper ... we felt that life was worth living after all ... there was no tin-opener to be found ... I took the tin off myself and hammered at it till I was worn out and sick at heart, whereupon Harris took it in hand. We beat it out flat; we beat it back square; we battered it into every form known to geometry — but we could not make a hole in it. Then George went at it, and knocked it into a shape, so strange, so weird, so unearthly in its wild hideousness, that he got frightened.”

Jerome K. Jerome

1.1 An overview

The first thing we must do is firmly put down any notion you may have that we’re about to start making maps. The only thing topology and topography have in common is their derivation from the Greek τόπος, or space. Topology is a relatively new field of mathematics and is related to geometry. In both of these subjects one studies the shape of things. In geometry, one characterizes, for example, a can of pineapple by its height, radius, surface area, and volume. In topology, one tries to identify the more subtle property that makes it impossible to get the pineapple out of the tin, no matter what shape it is battered into, as long as one does not puncture the can.

In a geometry course, one studies at first simple figures such as triangles or quadrilaterals and their properties, such as the lengths of the sides, the measures of the angles, and the areas enclosed. Next, one develops a definition of when figures are geometrically the same or congruent. Congruent figures will have equal geometric properties: lengths, angle measures, areas, etc. One can study the set of functions, called isometries or rigid motions (such as rotations), which preserve congruence (so that $f(\Delta)$ is congruent to $\Delta$). One develops theorems to tell when figures are congruent (side-angle-side congruence, etc.). One identifies the properties of certain classes of figures, such as the angle-sum theorem for triangles or the Pythagorean Theorem for right triangles.
We wish to emulate this course of study for topology and the more elusive properties not detected by geometry. Since length and angle measure are adequately covered in geometry, one decides to ignore these factors. Any two line segments, even of different lengths, are considered to be topologically equal, since by stretching one could be turned into the other. All angles are equal, since one could be bent to form the other. Thus, a square is the same topological shape as a rectangle or any other quadrilateral. By straightening one of the angles, a rectangle is seen to be the same as a triangle. One is also allowed to flex or bend lines into squiggles or curves. Thus, in topology, one considers the examples in each row of Figure 1.1 below to be same in topology, since they differ only in the geometric properties of length, angle measure, and curvature.

![Fig. 1.1. The figures in each row are topologically the same](image)

One concentrates, instead, on how a line segment and a circle differ. It seems obvious that the line and circle differ in some way not explicitly described by length or curvature or any other classical geometric property. The essence of this difference is the property that the circle or ring divides the plane into a region inside the circle and another region outside, but the line segment does not divide the plane. Even an infinite line, which does divide the plane, does so in a manner intrinsically different from the circle. We wish to quantify those properties in which line and the circle and similar figures differ and to define when objects are to be considered topologically equal and what properties are preserved by this type of equality.

Two objects are topologically identical if there is a continuous deformation (to be more precisely defined later) from one to the other. The bending and stretching that we allowed in Figure 1.1 are examples of such continuous deformations. Thus, topology is sometimes called the study of continuity, or, a more hackneyed term, "rubber-sheet geometry," as one pretends everything is formed of extremely flexible rubber. A mathematical cliché (included only in the interest of cultural literacy) is illustrated in Figure 1.2.