A. Introduction

By a sequential test of a statistical hypothesis is meant any statistical test procedure which gives a specific rule, at any stage of the experiment (at the $n$-th trial for each integral value of $n$), for making one of the following three decisions: (1) to accept the hypothesis being tested (null hypothesis), (2) to reject the null hypothesis, (3) to continue the experiment by making an additional observation. Thus, such a test procedure is carried out sequentially. On the basis of the first trial, one of the three decisions mentioned above is made. If the first or the second decision is made, the process is terminated. If the third decision is made, a second trial is performed. Again on the basis of the first two trials one of the three decisions is made and if the third decision is reached a third trial is performed, etc. This process is continued until either the first or the second decision is made.

An essential feature of the sequential test, as distinguished from the current test procedure, is that the number of observations required by the sequential test is not predetermined, but is a random variable due to the fact that at any stage of the experiment the decision of terminating the process depends on the results of the observations previously made. The current test procedure may be considered a limiting case of a sequential test in the following sense: For any positive integer $n$ less than some fixed positive integer $N$, the third decision is always taken at the $n$-th trial irrespective of the results of these first $n$ trials. At the $N$-th trial either the first or the second decision is taken. Which decision is taken will depend, of course, on the results of the $N$ trials.

In a sequential test, as well as in the current test procedure, we may commit two kinds of errors. We may reject the null hypothesis when it is true (error of the first kind), or we may accept the null hypothesis when some alternative
hypothesis is true (error of the second kind). Suppose that we wish to test the null hypothesis $H_0$ against a single alternative hypothesis $H_1$, and that we want the test procedure to be such that the probability of making an error of the first kind (rejecting $H_0$ when $H_0$ is true) does not exceed a preassigned value $\alpha$, and the probability of making an error of the second kind (accepting $H_0$ when $H_1$ is true) does not exceed a preassigned value $\beta$. Using the current test procedure, i.e., a most powerful test for testing $H_0$ against $H_1$ in the sense of the Neyman-Pearson theory, the minimum number of observations required by the test can be determined as follows: For any given number $N$ of observations a most powerful test is considered for which the probability of an error of the first kind is equal to $\alpha$. Let $\beta(N)$ denote the probability of an error of the second kind for this test procedure. Then the minimum number of observations is equal to the smallest positive integer $N$ for which $\beta(N) \geq \beta$.

In this paper a particular test procedure, called the sequential probability ratio test, is devised and shown to have certain optimum properties (see section 4.7). The sequential probability ratio test in general requires an expected number of observations considerably smaller than the fixed number of observations needed by the current most powerful test which controls the errors of the first and second kinds to exactly the same extent (has the same $\alpha$ and $\beta$) as the sequential test. The sequential probability ratio test frequently results in a saving of about 50% in the number of observations as compared with the current most powerful test. Another surprising feature of the sequential probability ratio test is that the test can be carried out without determining any probability distributions whatsoever. In the current procedure the test can be carried out only if the probability distribution of the statistic on which the test is based is known. This is not necessary in the application of the sequential probability ratio test, and only simple algebraic operations are needed for carrying it out. Distribution problems arise in connection with the sequential probability ratio test only if we want to make statements about the probability distribution of the number of observations required by the test.

This paper consists of two parts. Part I deals with the theory of sequential tests for testing a simple hypothesis against a single alternative. In Part II a theory of sequential tests for testing simple or composite hypotheses against infinite sets of alternatives is outlined. The extension of the probability ratio test to the case of testing a simple hypothesis against a set of one-sided alternatives is straightforward and does not present any difficulty. Applications to testing the means of binomial and normal distributions, as well as to testing double dichotomies are given. The theory of sequential tests of hypotheses with no restrictions on the possible values of the unknown parameters is, however, not as simple. There are several unsolved problems in this case and it is hoped that the general ideas outlined in Part II will stimulate further research.

Sections 5.2, 5.3 and 5.4 in Part II deal with the applications of the sequential probability ratio test to binomial distributions, double dichotomies and normal distributions. These sections are nearly self-contained and can be