Chapter 14

Definite Integrals and the Residue Calculus

The residue calculus is eminently suited to evaluating real integrals whose integrands have no known explicit antiderivatives. The basic idea is simple: The real interval of integration is incorporated into a closed path $\gamma$ in the complex plane and the integrand is then extended into the region bounded by $\gamma$. The extension is required to be holomorphic there except for isolated singularities. The integral over $\gamma$ is then determined from the residue theorem, and the needed residues are computed algebraically. Euler, Laplace and Poisson needed considerable analytic inventiveness to find their integrals. But today it would be more a question of proficiency in the use of the Cauchy formulas. Nevertheless there is no canonical method of finding, for a given integrand and interval of integration, the best path $\gamma$ in $\mathbb{C}$ to use.

We will illustrate the techniques with a selection of typical examples in sections 1 and 2, "but even complete mastery does not guarantee success" (AhlFors [1], p.154). In each case it is left to the reader to satisfy himself that the path of integration being employed is simply closed. In section 3 the Gauss sums will be evaluated residue-theoretically.

§1 Calculation of integrals

The examples assembled in this section are very simple. But everyone studying the subject should master the techniques of dealing with these
types of integrals – this circle of ideas is a popular source of examination
questions. First we are going to recall some simple facts from the theory of
improper integrals. For details we refer the reader to Edmund Landau’s
book Differential and Integral Calculus, Chelsea Publ. Co. (1950), New
York (especially Chapter 28).

0. Improper integrals. If \( f: [a, \infty) \to \mathbb{C} \) is continuous, then, as all
readers know, we set

\[
\int_a^\infty f(x)dx := \lim_{s \to \infty} \int_a^s f(x)dx
\]

whenever the limit on the right exists; \( \int_a^\infty f(x)dx \) is called an improper
integral. Calculations with such integrals obey some rather obvious rules,
e.g.,

\[
\int_a^\infty f(x)dx = \int_b^\infty f(x)dx + \int_a^b f(x)dx \quad \text{for all } b > a.
\]

Improper integrals of the form \( \int_{-\infty}^a f(x)dx \) are defined in a similar way. Finally we set

\[
\int_{-\infty}^\infty f(x)dx := \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx = \lim_{r,s \to \infty} \int_{-r}^s f(x)dx
\]

whenever \( f: \mathbb{R} \to \mathbb{C} \) is continuous and the two limits involved both exist.
It is to be emphasized that \( r \) and \( s \) have to be allowed to run to \( \infty \) inde­
pendently of each other; that is, the existence of \( \lim_{r \to \infty} \int_{-r}^s f(x)dx \) does
not imply the existence of \( \int_{-\infty}^\infty f(x)dx \) as we are defining the latter. The
function \( f(x) = x \) demonstrates this convincingly.

Basic to the theory of improper integrals is the following

**Existence criterion.** If \( f: [a, \infty) \to \mathbb{C} \) is continuous and there is a \( k > 1 \)
such that \( x^k f(x) \) is bounded then \( \int_a^\infty f(x)dx \) exists.

This follows rather easily from the Cauchy convergence criterion. The
hypothesis \( k > 1 \) is essential, since, for example, \( \int_2^\infty \frac{dx}{x \log x} \) does not exist
even though \( x(x \log x)^{-1} \to 0 \) as \( x \to \infty \). Also, though the handy word
“criterion” was used, the boundedness of \( x^k f(x) \) for some \( k > 1 \) is only
a sufficient and certainly not a necessary condition for the existence of
\( \int_a^\infty f(x)dx \). For example, both improper integrals

\[
\int_0^\infty \frac{\sin x}{x}dx , \quad \int_0^\infty \sin(x^2)dx
\]

exist, although there is no \( k > 1 \) such that either \( x^k \sin \frac{x}{x} \) or \( x^k \sin(x^2) \)
is bounded. In the second of these two examples the integrand does not