Chapter 7

The Integral Theorem, Integral Formula and Power Series Development

Integralsatz und Integralformel sind zusammen von solcher Tragweite, dass man ohne Uebertriebung sagen kann, in diesen beiden Integralen liege die ganze jetzige Functionentheorie concentrirt vor (The integral theorem and the integral formula together are of such scope that one can say without exaggeration: the whole of contemporary function theory is concentrated in these two integrals) — L. KRONECKER.

The era of complex integration begins with CAUCHY. It is consequently condign that his name is associated with practically every major result of this theory. In this chapter the principal Cauchy theorems will be derived in their simplest forms and extensively discussed (sections 1 and 2). We show in section 3 the most important application which is that holomorphic functions may be locally developed into power series. "Ceci marque un des plus grands progrès qui aient jamais été réalisés dans l’Analyse. (This marks one of the greatest advances that have ever been realized in analysis.)" — [Lin], pp. 9,10. As a consequence of the CAUCHY–TAYLOR development of a function we immediately prove (in 3.4) the Riemann continuation theorem, which is indispensable in many subsequent consid-
§1 The Cauchy Integral Theorem for star regions

The main result of this section is theorem 2. In order to prove it we will need in addition to integrability criterion 6.3.3, the

1. Integral lemma of GOURSAT. Let \( f \) be holomorphic in the domain \( D \). Then for the boundary \( \partial \Delta \) of every triangle \( \Delta \subset D \) we have

\[
\int_{\partial \Delta} f d\zeta = 0.
\]

For the proof we require two elementary facts about perimeters of triangles:

1) \( \max_{w, z \in \Delta} |w - z| \leq L(\partial \Delta) \).

2) \( L(\partial \Delta') = \frac{1}{2} L(\partial \Delta) \) for each of the four congruent sub-triangles \( \Delta' \) arising from connecting the midpoints of the three sides of \( \Delta \) (cf. the left-hand figure below).

We now prove the integral lemma. As a handy abbreviation we use \( a(\Delta) := \int_{\partial \Delta} f d\zeta \). By connecting with straight line segments the midpoints of the sides of \( \Delta \) we divide \( \Delta \) into four congruent sub-triangles \( \Delta_\nu \), \( 1 \leq \nu \leq 4 \); and then