This is the first of four lectures—§11–14—that comprise in some sense the heart of the book. In particular, the naive analysis of §11.1, together with the analogous parts of §12 and §13, form the paradigm for the study of finite-dimensional representations of all semisimple Lie algebras and groups. §11.2 is less central; in it we show how the analysis carried out in §11.1 can be used to explicitly describe the tensor products of irreducible representations. §11.3 is least important; it indicates how we can interpret geometrically some of the results of the preceding section. The discussions in §11.1 and §11.2 are completely elementary (we do use the notion of symmetric powers of a vector space, but in a non-threatening way). §11.3 involves a fair amount of classical projective geometry, and can be skimmed or skipped by those not already familiar with the relevant basic notions from algebraic geometry.

§11.1: The irreducible representations
§11.2: A little plethysm
§11.3: A little geometric plethysm

§11.1. The Irreducible Representations

We start our discussion of representations of semisimple Lie algebras with the simplest case, that of $\mathfrak{sl}_2 \mathbb{C}$. As we will see, while this case does not exhibit any of the complexity of the more general case, the basic idea that informs the whole approach is clearly illustrated here.

This approach is one already mentioned above, in connection with the representations of the symmetric group on three letters. The idea in that case was that given a representation of our group on a vector space $V$ we first restrict the representation to the abelian subgroup generated by a 3-cycle $\tau$. We obtain a decomposition
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\[ V = \bigoplus V_\alpha \]

of \( V \) into eigenspaces for the action of \( \tau \); the commutation relations satisfied by the remaining elements \( \sigma \) of the group with respect to \( \tau \) implied that such \( \sigma \) simply permuted these subspaces \( V_\alpha \), so that the representation was in effect determined by the collection of eigenvalues of \( \tau \).

Of course, circumstances in the case of Lie algebra representations are quite different: to name two, it is no longer the case that the action of an abelian object on any vector space admits such a decomposition; and even if such a decomposition exists we certainly cannot expect that the remaining elements of our Lie algebra will simply permute its summands. Nevertheless, the idea remains essentially a good one, as we shall now see.

To begin with, we choose the basis for the Lie algebra \( \mathfrak{sl}_2 \mathbb{C} \) from the last lecture:

\[ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

satisfying

\[ [H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \quad (11.1) \]

These seem like a perfectly natural basis to choose, but in fact the choice is dictated by more than aesthetics; there is, as we will see, a nearly canonical way of choosing a basis of a semisimple Lie algebra (up to conjugation), which will yield this basis in the present circumstance and which will share many of the properties we describe below.

In any event, let \( V \) be an irreducible finite-dimensional representation of \( \mathfrak{sl}_2 \mathbb{C} \). We start by trotting out one of the facts that we quoted in Lecture 9, the preservation of Jordan decomposition; in the present circumstances it implies that

\textit{The action of} \( H \) \textit{on} \( V \) \textit{is diagonalizable.} \quad (11.2)

We thus have, as indicated, a decomposition

\[ V = \bigoplus V_\alpha, \quad (11.3) \]

where the \( \alpha \) run over a collection of complex numbers, such that for any vector \( v \in V_\alpha \) we have

\[ H(v) = \alpha \cdot v. \]

The next question is obviously how \( X \) and \( Y \) act on the various spaces \( V_\alpha \). We claim that \( X \) and \( Y \) must each carry the subspaces \( V_\alpha \) into other subspaces \( V_\beta \). In fact, we can be more specific: if we want to know where the image of a given vector \( v \in V_\alpha \) under the action of \( X \) sits in relation to the decomposition (11.3), we have to know how \( H \) acts on \( X(v) \); this is given by the