LECTURE 3

Examples; Induced Representations; Group Algebras; Real Representations

This lecture is something of a grabbag. We start in §3.1 with examples illustrating the use of the techniques of the preceding lecture. Section 3.2 is also by way of an example. We will see quite a bit more about the representations of the symmetric groups in general later; §4 is devoted to this and will certainly subsume this discussion, but this should provide at least a sense of how we can go about analyzing representations of a class of groups, as opposed to individual groups. In §§3.3 and 3.4 we introduce two basic notions in representation theory, induced representations and the group algebra. Finally, in §3.5 we show how to classify representations of a finite group on a real vector space, given the answer to the corresponding question over \( \mathbb{C} \), and say a few words about the analogous question for subfields of \( \mathbb{C} \) other than \( \mathbb{R} \). Everything in this lecture is elementary except Exercises 3.9 and 3.32, which involve the notions of Clifford algebras and the Fourier transform, respectively (both exercises, of course, can be skipped).

§3.1: Examples: \( S_5 \) and \( A_5 \)
§3.2: Exterior powers of the standard representation of \( S_4 \)
§3.3: Induced representations
§3.4: The group algebra
§3.5: Real representations and representations over subfields of \( \mathbb{C} \)

§3.1. Examples: \( S_5 \) and \( A_5 \)

We have found the representations of the symmetric and alternating groups for \( n \leq 4 \). Before turning to a more systematic study of symmetric and alternating groups, we will work out the next couple of cases.
§3.1. Examples: $\mathfrak{S}_5$ and $\mathfrak{A}_5$

Representations of the Symmetric Group $\mathfrak{S}_5$

As before, we start by listing the conjugacy classes of $\mathfrak{S}_5$ and giving the number of elements of each: we have 10 transpositions, 20 three-cycles, 30 four-cycles and 24 five-cycles; in addition, we have 15 elements conjugate to $(12)(34)$ and 10 elements conjugate to $(12)(345)$. As for the irreducible representations, we have, of course, the trivial representation $U$, the alternating representation $U'$, and the standard representation $V$; also, as in the case of $\mathfrak{S}_4$ we can tensor the standard representation $V$ with the alternating one to obtain another irreducible representation $V'$ with character $\chi_{V'} = \chi_V \cdot \chi_{U'}$.

**Exercise 3.1.** Find the characters of the representations $V$ and $V'$; deduce in particular that $V$ and $V'$ are distinct irreducible representations.

The first four rows of the character table are thus

<table>
<thead>
<tr>
<th>$\mathfrak{S}_5$</th>
<th>1</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>24</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$U'$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$V$</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$V'$</td>
<td>4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Clearly, we need three more irreducible representations. Where should we look for these? On the basis of our previous experience (and Problem 2.37), a natural place would be in the tensor products/powers of the irreducible representations we have found so far, in particular in $V \otimes V$ (the other two possible products will yield nothing new: we have $V' \otimes V = V \otimes V \otimes U'$ and $V' \otimes V' = V \otimes V$). Of course, $V \otimes V$ breaks up into $\Lambda^2 V$ and $\text{Sym}^2 V$, so we look at these separately. To start with, by the formula

$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

we calculate the character of $\Lambda^2 V$:

$$\chi_{\Lambda^2 V} = (6, 0, 0, 0, 1, -2, 0);$$

we see from this that it is indeed a fifth irreducible representation (and that $\Lambda^2 V \otimes U' = \Lambda^2 V$, so we get nothing new that way).

We can now find the remaining two representations in either of two ways. First, if $n_1$ and $n_2$ are their dimensions, we have

$$5! = 120 = 1^2 + 1^2 + 4^2 + 4^2 + 6^2 + n_1^2 + n_2^2,$$

so $n_1^2 + n_2^2 = 50$. There are no more one-dimensional representations, since these are trivial on normal subgroups whose quotient group is cyclic, and $\mathfrak{A}_5$