The transition from (what is here called) the mechanics of Lagrange, with time as the main parameter, to the mechanics of Hamilton and Jacobi, with energy as the controlling variable, is formally easy to carry out. Its importance becomes apparent when one tries to solve special problems. The test case is the motion of a body where the force decreases as the inverse square of the distance from the origin. It will be treated at the end of this chapter and will be given an appealing geometrical solution.

2.1 Phase Space and Its Hamiltonian

The state of a dynamical system at the time $t$ is now specified by giving its momentum $p$ and its position $q$, rather than its velocity $\dot{q}$ and its position $q$. If we start by describing the system with the help of its Lagrangian $L$, then we have to make the transition from the velocity $\dot{q}$ to the momentum $p$ via the formula (1.1). This kind of transformation is well known from thermodynamics and is generally called a Legendre transformation. It implies a change in the function describing the system at the same time as using its derivatives as the new variables; in our case that means transforming the Lagrangian $L$, which is a function of $\dot{q}, q,$ and $t$, into the Hamiltonian $H$, which is a function of $p, q,$ and $t$, with the help of the formulas
Comparing with (1.7), the Hamiltonian can be interpreted as the energy of the dynamical system at the time $t$.

The space whose points are defined by the $n$ momenta $p$ and the $n$ coordinates $q$ is called the phase space of the dynamical system. For the familiar Lagrangian $L = T - V$ with $T = \frac{1}{2}m\dot{q}^2$ and $V$ depending only on $q$, one finds $H = T + V$ with $T = \frac{p^2}{2m}$. In mathematical terminology this is the cotangent bundle for the manifold in which the dynamical system moves.

The Hamilton-Jacobi equations of motion (1.2) become

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \quad \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j},$$

(2.2)
two sets of first-order equations, rather than one set of second-order equations in the Newtonian tradition. We can regard them as defined by a vector-field in phase space, which defines a flow in phase space. It is given by some kind of gradient of the Hamiltonian $H$. The opposite signs in the two sets of equations are crucial and cannot be eliminated by any simple device. The consequences of these differing signs are pursued in a special discipline, symplectic geometry, which is obviously important to the study of mechanics; but we will not discuss this field, except for a short excursion in Chapter 7.

Conservation of energy in Lagrangian mechanics follows from the fact that the Lagrangian $L$ does not depend explicitly on the time $t$; in complete analogy, conservation of energy in Hamiltonian mechanics requires that $\frac{\partial H}{\partial t} = 0$. The value of $H(p, q)$ then remains constant along any trajectory. This constant value is usually designated by $E$ as before, so that we will write $H(p, q) = E$. In a large measure, the value of $E$ for a particular trajectory will replace the parameter $t$ in many applications. The duality between $E$ and $t$ comes out quite naturally in Hamiltonian mechanics, but its full significance can only be appreciated in quantum mechanics.

### 2.2 The Action Function $S$

The replacement of $t$ by $E$ as the independent parameter requires that we use a new kind of action integral $S(q''q'q'qE)$. It is again defined by a trajectory from $q'$ to $q''$; but instead of the given time interval $t$, the energy $E$ of the trajectory is now stipulated. In doing so, another Legendre transformation is made,