Solutions with Shocks for Conservation Laws

with Memory

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0. Motivation

The equations of motion of a one-dimensional body with unit reference density and zero body force, in Lagrangian coordinates, read

\begin{align}
\frac{\partial}{\partial t} u(x,t) - \frac{\partial}{\partial x} v(x,t) &= 0 \\
\frac{\partial}{\partial t} v(x,t) - \frac{\partial}{\partial x} \sigma(x,t) &= 0
\end{align}

(0.1)

where \( u \) is deformation gradient, \( v \) is velocity, and \( \sigma \) denotes stress.

When the body is elastic, the stress at the material point \( x \) and time \( t \) is determined solely by the value of deformation gradient at \( (x,t) \) via a constitutive relation

\[ \sigma(x,t) = f(u(x,t)) \]

(0.2)

Under the standard assumption \( f'(u) > 0 \), (0.1), (0.2) yield a strictly hyperbolic system for which the Cauchy problem has been studied extensively: when the initial data are smooth, a classical solution starts out at \( t = 0 \) but eventually breaks down in a finite time, with the formation of shocks (cf. [14, 13]). When the initial data have small
total variation, an admissible weak solution exists, globally in time, in the class BV of functions of bounded variation (cf. [9]). Furthermore, if the system is genuinely nonlinear, in the sense that $f''(u) \neq 0$ on its domain of definition, then a globally defined admissible BV solution exists under initial data that are merely bounded measurable and have small oscillation (cf. [10]). There is strong evidence that globally defined weak solutions exist in the class of bounded measurable functions, under any bounded measurable initial data, though this has been strictly established only when $(u - \bar{u})f''(u) \geq 0$ for some $\bar{u}$ and all $u$ (cf. [7, 20]).

It is of interest to compare and contrast the behavior of elastic bodies with the behavior of materials with fading memory (cf. [22]), in which the stress $\sigma(x,t)$ at the material point $x$ and time $t$ is determined by the entire history $u^{(t)}(x, \cdot)$ of deformation gradient at $x$, defined by $u^{(t)}(x, \tau) = u(x, t-\tau)$, $0 \leq \tau < \infty$, through a functional with appropriate smoothness properties.

For concreteness, let us consider the simple model

\begin{equation}
\sigma(x, t) = f(u(x, t)) - z(x, t) \tag{0.3}
\end{equation}

where

\begin{equation}
z(x, t) = \int_{-\infty}^{t} K(t-\tau) g(u(x, \tau)) d\tau . \tag{0.4}
\end{equation}

The study of acceleration wave propagation in media of this type (cf. [11]) suggests that the memory exerts a rather weak damping influence. This is corroborated by the observation (cf. [15]) that when $g(u) = f(u)$ the system (0.1), (0.3), (0.4) may be rewritten in the form

\begin{equation}
\begin{cases}
\partial_t u(x, t) - \partial_x v(x, t) = 0 \\
\partial_t v(x, t) - \partial_x f(u(x, t)) + K(0) v(x, t) - \int_{-\infty}^{t} L'(t-\tau) v(x, \tau) d\tau = 0 ,
\end{cases} \tag{0.5}
\end{equation}