CHAPTER XVII

The Change of Variables Formula

If you have not already done so, you should now read the section on cross products, Chapter I, §7 because we are going to use it.

XVII, §1. DETERMINANTS AS AREA AND VOLUME

We shall study the manner in which area changes under an arbitrary mapping by approximating this mapping with a linear map. Therefore, first we study how area and volume change under a linear map, and this leads us to interpret the determinant as area and volume according as we are in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

Let us first consider \( \mathbb{R}^2 \). Let

\[
A = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b \\ d \end{pmatrix}
\]

be two non-zero vectors in the plane, and suppose that they are not scalar multiples of each other. We have already seen that they span a parallelogram, as shown on Fig. 1.

![Figure 1](image-url)
Theorem 1.1 in $\mathbb{R}^2$. Let $A$, $B$ be non-zero elements of $\mathbb{R}^2$, which are not scalar multiples of each other. Then the area of the parallelogram spanned by $A$ and $B$ is equal to the absolute value of the determinant $|D(A, B)|$.

Proof. We assume known that this area is equal to the product of the lengths of the base times the altitude, and this is equal to

$$\|A\| \|B\| |\sin \theta|,$$

where $\theta$ is the angle between $A$ and $B$ (i.e. between $\overrightarrow{OA}$ and $\overrightarrow{OB}$). This is illustrated on Fig. 2.

![Figure 2](image)

Note that

$$|\sin \theta| = \sqrt{1 - \cos^2 \theta},$$

and recall from the theory of the dot product that

$$\cos \theta = \frac{A \cdot B}{\|A\| \|B\|}.$$

We have

$$\text{Area of parallelogram} = \|A\| \|B\| \sqrt{1 - \frac{(A \cdot B)^2}{\|A\|^2 \|B\|^2}}$$

$$= \sqrt{\|A\|^2 \|B\|^2 - (A \cdot B)^2}.$$  

All that remains to be done is to plug in the coordinates of $A$ and $B$ to see what we want come out. Indeed, the above expression is equal to the square root of

$$(a^2 + c^2)(b^2 + d^2) - (ab + cd)^2.$$  

If you expand this out, you will find that this last expression is equal to

$$(ad - bc)^2.$$