Bivariate continuous distributions are defined in Section 1, and change of variables problems are considered in Section 2. In Section 3, we prove some results which will be needed in deriving statistical methods for analyzing normally distributed measurements. Sections 4 and 5 deal with properties and applications of the bivariate normal distribution. The discussion and examples are mostly confined to the two-variable case, but the extension to multivariate distributions is straightforward.

7.1. Definitions and Notation

The joint cumulative distribution function $F$ of two real-valued variates $X$ and $Y$ was defined in Section 4.5. $F$ is a function of two variables:

$$F(x,y) = P(X \leq x, Y \leq y) \quad \text{for all real } x, y.$$  \hfill (7.1.1)

Suppose that $F(x,y)$ is continuous, and that the derivative

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$  \hfill (7.1.2)

exists and is continuous (except possibly along a finite number of curves). Furthermore, suppose that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = 1.$$  \hfill (7.1.3)

Then $X$ and $Y$ are said to have a \textit{bivariate continuous distribution}, and $f$ is called their \textit{joint probability density function}.
Condition (7.1.3) is needed to rule out the possibility of a concentration of probability in a one-dimensional subspace of the \((x, y)\) plane. For instance, if half the probability is spread uniformly over the unit square and the other half is spread uniformly over its main diagonal \(x = y\), conditions (7.1.1) and (7.1.2) are satisfied, but (7.1.3) is not (see Problem 7.1.8). In the univariate case, the continuity of \(F\) is enough to ensure that there is not a concentration of probability at any single point (0-dimensional subspace).

The joint p.d.f. is a non-negative function. The probability that \((X, Y)\) belongs to a region \(R\) in the \((x, y)\) plane is given by the volume under the surface \(z = f(x, y)\) above the region \(R\); that is,

\[
P\{(X, Y) \in R\} = \int_R \int f(x, y) \, dx \, dy. \tag{7.1.4}
\]

In particular, we have

\[
F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} ds \int_{-\infty}^{y} f(s, t) \, dt. \tag{7.1.5}
\]

The marginal c.d.f. of \(X\) is the c.d.f. of \(X\) alone:

\[
F_1(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = \int_{-\infty}^{x} ds \int_{-\infty}^{\infty} f(s, y) \, dy. \tag{7.1.6}
\]

Differentiating with respect to \(x\) gives the marginal p.d.f. of \(X\):

\[
f_1(x) = \frac{d}{dx}F_1(x) = \int_{-\infty}^{\infty} f(x, y) \, dy. \tag{7.1.7}
\]

The marginal p.d.f. of \(X\) can thus be found by integrating the unwanted variable \(Y\) out of the joint p.d.f. The marginal c.d.f. and p.d.f. of \(Y\) may be defined similarly.

**Example 7.1.1.** Let \(X\) and \(Y\) be continuous variates with joint p.d.f.

\[
f(x, y) = \begin{cases} k(x^2 + 2xy) & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}
\]

(a) Evaluate the normalizing constant \(k\).
(b) Find the joint c.d.f. of \(X\) and \(Y\) and the marginal p.d.f. of \(X\).
(c) Compute the probability of the event \(\{X \leq Y\}\).

**Solution.** (a) The total volume under the p.d.f. must be 1, and hence

\[
1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{0}^{1} dx \int_{0}^{1} (x^2 + 2xy) \, dy
\]

\[
= k \int_{0}^{1} dx \left[ x^2 y + xy^2 \right]_{y=0}^{y=1} = k \int_{0}^{1} (x^2 + x) \, dx
\]

\[
= k \left[ \frac{x^3}{3} + \frac{x^2}{2} \right]_{0}^{1} = \frac{5k}{6}.
\]

Therefore \(k = \frac{6}{5}\).