§1. Introduction to Radon Measures on Hausdorff Spaces

It is well known that the pure set-theoretical theory of measure and integration has its limitations, and many interesting results need a topological frame because measure spaces without an underlying "nice" topological structure may be very pathological. In classical analysis this difficulty was overcome by introducing the theory of Radon measures on locally compact spaces. On these spaces there is a particularly important one-to-one relationship between Radon measures and certain linear functionals (see below) which in many treatments on analysis leads to the definition, that a Radon measure is a linear functional with certain properties.

Another branch of mathematics with a need for a highly developed measure theory is probability theory. Here the class of locally compact spaces turned out to be far too narrow, partly due to the fact that an infinite dimensional topological vector space never can be locally compact. For example, it was found that the class of polish spaces (i.e. separable and completely metrizable spaces) was much more appropriate for probabilistic purposes.

Later on it became clear that a very satisfactory theory of Radon measures can be developed on arbitrary Hausdorff spaces. This has been done, for example, in L. Schwartz's monograph (1973). We shall follow an approach to Radon measure theory which has been initiated by Kisyński and developed by Topsøe. It deviates, for example, from the Schwartz–Bourbaki theory in working only with inner approximation, but we hope to show that it gives an easy and elegant access to the main results.
In the following let \( X \) denote an arbitrary Hausdorff space. The natural \( \sigma \)-algebra on which the measures considered will be defined will always be the \( \sigma \)-algebra \( \mathcal{B} = \mathcal{B}(X) \) of all Borel subsets of \( X \), i.e. the \( \sigma \)-algebra generated by the open subsets of \( X \). In our terminology a measure will always be non-negative; a measure defined on \( \mathcal{B}(X) \) will be called a Borel measure on \( X \). Later on we also need to consider \( \sigma \)-additive functions on \( \mathcal{B}(X) \) which may assume negative values, these functions will be called signed measures.

1.1. Definition. A Radon measure \( \mu \) on the Hausdorff space \( X \) is a Borel measure on \( X \) satisfying

(i) \( \mu(C) < \infty \) for each compact subset \( C \subseteq X \),
(ii) \( \mu(B) = \sup\{\mu(C) | C \subseteq B, C \text{ compact}\} \) for each \( B \in \mathcal{B}(X) \).

The set of all Radon measures on \( X \) is denoted \( M_+(X) \).

Remark. Many authors require a Radon measure to be locally finite, i.e. each point has an open neighbourhood with finite measure. There are good reasons for not having this condition as part of the definition, see Notes and Remarks at the end of this chapter. A finite Radon measure \( \mu \) (i.e. \( \mu(X) < \infty \)) satisfies
\[
\mu(B) = \inf\{\mu(G) | B \subseteq G, G \text{ open}\}
\]
for \( B \in \mathcal{B}(X) \) as is easily seen by considering the property (ii) for \( B \). However for arbitrary Radon measures this need not be true as is shown by Exercise 1.30 below.

Let \( \mathcal{K} = \mathcal{K}(X) \) denote the family of all compact subsets of \( X \). Clearly the restriction to \( \mathcal{K} \) of a Radon measure \( \mu \) is a set function

\[\lambda: \mathcal{K} \rightarrow [0, \infty[\]

satisfying the axioms of a Radon content below.

1.2. Definition. A Radon content is a set function \( \lambda: \mathcal{K} \rightarrow [0, \infty[ \) which satisfies
\[
\lambda(C_2) - \lambda(C_1) = \sup\{\lambda(C) | C \subseteq C_2 \setminus C_1, C \in \mathcal{K}\}
\]
for all \( C_1, C_2 \in \mathcal{K} \) with \( C_1 \subseteq C_2 \).

The key result in our approach to Radon measure theory is the extension theorem (1.4) below, the proof of which will need the following lemma.

1.3. Lemma. A Radon content \( \lambda \) has the following properties:

(i) \( \lambda(C_1) \leq \lambda(C_2) \) for all \( C_1, C_2 \in \mathcal{K}, C_1 \subseteq C_2 \), i.e. \( \lambda \) is monotone.
(ii) \( \lambda(C_1 \cup C_2) + \lambda(C_1 \cap C_2) = \lambda(C_1) + \lambda(C_2) \), i.e. \( \lambda \) is modular.
(iii) If a net \( (C_a)_{a \in A} \) in \( \mathcal{K} \) is decreasing with \( C = \bigcap_{a \in A} C_a \) then \( \lambda(C) = \lim_{a} \lambda(C_a) = \inf_{a} \lambda(C_a) \). In particular for a decreasing sequence \( C_1 \supseteq C_2 \supseteq \cdots \) of compact sets we have \( \lim_{n \to \infty} \lambda(C_n) = \lambda(\bigcap_{n=1}^{\infty} C_n) \).