CHAPTER 3
General Results on Positive and Negative Definite Matrices and Kernels

§1. Definitions and Some Simple Properties of Positive and Negative Definite Kernels

When dealing with positive and negative definite kernels a certain amount of confusion often arises concerning terminology. A positive definite kernel defined on a finite set is usually called a positive semidefinite matrix. Sometimes it is only called "positive", which may be misleading. When working on groups, the name positive definite function is used traditionally. In our previous papers on abelian semigroups we also followed this tradition. Instead of calling a kernel $\psi$ negative definite, some authors call the kernel $-\psi$ "conditionally positive definite" or "almost positive." In this book we use mainly the larger class of "semidefinite" kernels of all kinds and therefore prefer to avoid the prefix "semi" which otherwise would appear several hundred times.

Adapting the above point of view, an $n \times n$ matrix $A = (a_{jk})$ of complex numbers is called positive definite if and only if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} a_{jk} \geq 0$$

for all $\{c_1, \ldots, c_n\} \subseteq \mathbb{C}$.

It is well known that this is the case if and only if $A$ is hermitian (i.e. $a_{jk} = \overline{a_{kj}}$ for $j, k = 1, \ldots, n$) and the eigenvalues of $A$ are all $\geq 0$.

Similarly $A$ is called negative definite if and only if $A$ is hermitian and

$$\sum_{j,k=1}^{n} c_j \overline{c_k} a_{jk} \leq 0$$
for all \( \{c_1, \ldots, c_n\} \subseteq \mathbb{C} \) with the extra condition \( \sum_{j=1}^{n} c_j = 0 \). (This definition requires \( n \geq 2 \). Any \( 1 \times 1 \) matrix \( A = (a_{11}) \) with real \( a_{11} \) is called negative definite.)

1.1. **Definition.** Let \( X \) be a nonempty set. A function \( \varphi: X \times X \to \mathbb{C} \) is called a **positive definite kernel** if and only if

\[
\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(x_j, x_k) \geq 0
\]

for all \( n \in \mathbb{N} \), \( \{x_1, \ldots, x_n\} \subseteq X \) and \( \{c_1, \ldots, c_n\} \subseteq \mathbb{C} \). We call the function \( \varphi \) a **negative definite kernel** if and only if it is hermitian (i.e. \( \varphi(y, x) = \overline{\varphi(x, y)} \) for all \( x, y \in X \)) and

\[
\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(x_j, x_k) \leq 0
\]

for all \( n \geq 2 \), \( \{x_1, \ldots, x_n\} \subseteq X \) and \( \{c_1, \ldots, c_n\} \subseteq \mathbb{C} \) with \( \sum_{j=1}^{n} c_j = 0 \).

If the above inequalities are strict whenever \( x_1, \ldots, x_n \) are different and at least one of the \( c_1, \ldots, c_n \) does not vanish, then the kernel \( \varphi \) is called **strictly positive** (resp. **strictly negative**) definite.

1.2. **Remark.** In the above definitions it is enough to consider mutually different elements \( x_1, \ldots, x_n \in X \). In fact, if \( x_1, \ldots, x_n \in X \) are arbitrary and \( x_1', \ldots, x_p \) are the mutually different elements among the \( x_i \)'s, then

\[
\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(x_j, x_k) = \sum_{j,k=1}^{p} d_j d_k \varphi(x_{x_j}, x_{x_k}),
\]

where

\[
d_k := \sum_{i \mid x_i = x_{x_k}} c_i, \quad k = 1, \ldots, p.
\]

Furthermore, if \( \sigma: X \to X \) is a bijection, then \( \varphi \) is a positive (resp. negative) definite kernel if and only if \( \varphi \circ (\sigma \times \sigma) \) is a positive (resp. negative) definite kernel.

If \( X \) is a finite set, say \( X = \{x_1, \ldots, x_n\} \), then plainly \( \varphi \) is positive (resp. negative) definite if and only if the \( n \times n \) matrix

\[
(\varphi(x_j, x_k))_{1 \leq j,k \leq n}
\]

is positive (resp. negative) definite.

We now list some simple properties and examples of positive and negative definite kernels.

1.3. A kernel \( \varphi \) on \( X \times X \) is positive (resp. negative) definite if and only if for every finite subset \( X_0 \subseteq X \) the restriction of \( \varphi \) to \( X_0 \times X_0 \) is positive (resp. negative) definite.