Chapter 3

Linear Transformations and Matrices

In order to compare different mathematical systems of the same type, it is essential to study the functions from one system to another which preserve the operations of the system. Thus in calculus we study functions \( f \) which preserve the limit operation: if \( \lim_{i \to \infty} x_i = x \), then \( \lim_{i \to \infty} f(x_i) = f(x) \). These are the continuous functions. For vector spaces, we investigate functions that preserve the vector space operations of addition and scalar multiplication. These are the linear transformations. In this chapter we develop the language of linear transformations and the connection between linear transformations and matrices. Deeper results about linear transformations appear later.

11. LINEAR TRANSFORMATIONS

Linear transformations are certain kinds of functions, and it is a good idea to begin this section by reviewing some of the definitions and terminology of functions on sets.

(11.1) DEFINITION. Let \( X \) and \( Y \) be sets. A function \( f: X \to Y \) (sometimes called a mapping) is a rule which assigns to each element of a set \( X \) a unique element of a set \( Y \). We write \( f(x) \) for the element of \( Y \) which the function \( f \) assigns to \( x \). The set \( X \) is called the domain or domain of definition of \( f \). Two functions \( f \) and \( f' \) are said to be equal, and we write...
\( f = f', \) if they have the same domain \( X \) and if, for all \( x \in X, f(x) = f'(x). \) The function \( f \) is said to be one-to-one if \( x_1 \neq x_2 \) in \( X \) implies that \( f(x_1) \neq f(x_2). \) [Note that this is equivalent to the statement that \( f(x_1) = f(x_2) \) implies \( x_1 = x_2. \) The function \( f \) is said to be onto \( Y \) if every \( y \in Y \) can be expressed in the form \( y = f(x) \) for some \( x \in X; \) we shall say that \( f \) is a function of \( X \) into \( Y \) when we want to allow the possibility that \( f \) is not onto \( Y. \) A one-to-one function \( f \) of a set \( X \) onto a set \( Y \) is called a one-to-one correspondence of \( X \) onto \( Y. \)

Examples of functions are familiar from calculus. We give some examples to show that the conditions of being onto or one-to-one are independent of each other. For example, the function \( f: \mathbb{R} \to \mathbb{R} \) defined by
\[
 f(x) = x^2, \quad x \in \mathbb{R},
\]
is a function that is neither one-to-one nor onto (why?). The function \( g: \mathbb{R} \to \mathbb{R} \) defined by
\[
 g(x) = e^x
\]
is one-to-one [because \( g'(x) > 0 \) for all \( x, \) so that the function is strictly increasing in the sense that \( x_1 < x_2 \) implies \( g(x_1) < g(x_2) \).] But \( g \) is not onto, since \( g(x) > 0 \) for all real numbers \( x. \) The function \( h: \mathbb{R}^* \to \mathbb{R} \) defined by
\[
 h(x) = \log_e |x|, \quad x \in \mathbb{R}^* = \{x \in \mathbb{R}, \ x \neq 0\}
\]
is onto, but not one-to-one, since \( h(x) = h(-x) \) for all \( x. \)

We are now ready to define linear transformations from one vector space to another. Throughout this section, \( F \) denotes an arbitrary field.

\section*{(11.2) Definition.} Let \( V \) and \( W \) be vector spaces over \( F. \) A linear transformation of \( V \) into \( W \) is a function \( T: V \to W \) which assigns to each vector \( v \in V \) a unique vector \( w = T(v) \in W \) such that
\[
 T(v_1 + v_2) = T(v_1) + T(v_2), \quad v_1 \in V, \\
 T(\alpha v) = \alpha T(v), \quad \alpha \in F, \quad v \in V,
\]

We shall often use the notation \( Tv \) instead of \( T(v), \) in order to avoid a forest of parentheses. We shall also use the arrow notation rather freely, in statements such as "let \( f \) be a function of \( V \) into \( W.\)" or "the function \( f: V \to W \) carries \( v_1 \to w_1\)."

We first consider some examples of linear transformations.

\section*{Example A.} The function \( T: \mathbb{R}_1 \to \mathbb{R}_1 \) defined by
\[
 T(x) = ax
\]