Three-Way and Higher-Order Nested Classifications

7.0 PREVIEW

The results of the preceding chapter can be readily extended to the case of three-way and the general q-way nested or hierarchical classifications. As an example of a three-way nested classification, suppose a chemical company wishes to examine the strength of a certain liquid chemical. The chemical is made in large vats and then is barreled. To study the strength of the chemical, an analyst randomly selects three different vats of the product. Three barrels are selected at random from each vat and then three samples are taken from each barrel. Finally, two independent measurements are made on each sample. The physical layout can be depicted schematically as shown in Figure 7.1. In this experiment, barrels are nested within the levels of the factor vats and samples are nested within the levels of the factor barrels. This is the so-called three-way nested classification having two replicates or measurements. In this chapter, we consider the three-way nested classification and indicate its generalization to higher-order nested classifications.

7.1 MATHEMATICAL MODEL

Consider three factors $A$, $B$, and $C$ having $a$, $b$, and $c$ levels respectively. The $b$ levels of factor $B$ are nested under each level of $A$ and $c$ levels of factor $C$ are nested under each level of factor $B$ (within $A$), and there are $n$ replicates within the combination of levels of $A$, $B$, and $C$. The analysis of variance model for this type of experimental layout is taken as

\[ y_{ijk\ell} = \mu + \alpha_i + \beta_j(i) + \gamma_k(ij) + \epsilon_{i\ell(jk)} \]

where $\mu$ is the general mean, $\alpha_i$ is the effect due to the $i$-th level of factor $A$, $\beta_j(i)$ is the effect due to the $j$-th level of factor $B$ within the $i$-th level of factor $A$, $\gamma_k(ij)$ is the effect due to the $k$-th level of factor $C$ within the $i$-th level of factor $A$ and the $j$-th level of factor $B$, and $\epsilon_{i\ell(jk)}$ is the random error term for the $(i\ell(jk))$th replicate.
\[ \beta_{j(i)} \text{ is the effect due to the } j\text{-th level of factor } B \text{ within the } i\text{-th level of factor } A, \]
\[ \gamma_{k(ij)} \text{ is the effect due to the } k\text{-th level of factor } C \text{ within the } j\text{-th level of factor } B \text{ and the } i\text{-th level of factor } A, \]
\[ e_{l(ijk)} \text{ is the error term that represents the variation within each cell.} \]

When all the factors have systematic effects, Model I is applicable to the data in (7.1.1). When all the factors are random, Model II is appropriate; and when some factors are fixed and others are random, a mixed model or Model III is the appropriate one. The assumptions under Models I, II, and III are exact parallels to that of the model (6.1.1). For example, if we assume that \( A \) is fixed and \( B \) and \( C \) are random, then \( \alpha_i \)'s are unknown fixed constants with the restriction that \( \sum_{i=1}^{a} \alpha_i = 0 \), and \( \beta_{j(i)} \)'s, \( \gamma_{k(ij)} \)'s, and \( e_{l(ijk)} \)'s are mutually and completely uncorrelated random variables with zero means and variances \( \sigma_{\beta}^2 \), \( \sigma_{\gamma}^2 \), and \( \sigma_{e}^2 \) respectively.

### 7.2 ANALYSIS OF VARIANCE

The calculations of the sums of squares and the analysis of variance for the three-way nested design are similar to the analysis for the two-way nested design presented in Chapter 6. The formulae for the sums of squares together with their computational forms are simple extensions of the formulae for the two-way nested design given in Section 6.7. Thus, starting with the identity

\[ y_{ijkl} - \bar{y}_{..} = (\bar{y}_{i..} - \bar{y}_{..}) + (\bar{y}_{ij.} - \bar{y}_{i..}) + (\bar{y}_{ijk.} - \bar{y}_{ij.}) + (\bar{y}_{ijkl} - \bar{y}_{ijk.}), \]

the total sum of squares is partitioned as

\[ SS_T = SS_A + SS_{B(A)} + SS_{C(B)} + SS_E, \]