Chapter 7

Representation of the $q$-Deformed Oscillator

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ABSTRACT This paper is the continuation of a presentation delivered at the Fifth Symposium of the Chilean Mathematical Society and provides a complement to a recent paper of the author (see [2]) on the representations of a $q$-oscillator algebra.

In the usual formal definition, one goes from the $q$-Heisenberg algebra to the $q$-oscillator algebra by adding to the pair $(a, a^+)$ a self-adjoint operator $N$ satisfying the relations


Here we will add to a triple $(E, T, V)$ a one-parameter unitary group $(U_t)$ (formally $U_t = e^{itN}$) satisfying the relations

$$U_t T U_t^{-1} = T, \quad U_t V U_t^{-1} = e^{-itV}.$$

For simplicity we always assume $q > 1$.

1 Main result

Theorem 1.1. Every irreducible quadruple $(E, T, V, (U_t))$ is equivalent to a unique quadruple of the one parameter family $(E_{\gamma}, T_{\gamma}, V_{\gamma}, (U_t)_{\gamma})$, $\gamma \in \mathbb{R}$, where $E_{\gamma} = l^2(N)$, $T_{\gamma}e_n = [n]_q e_n$, $V_{\gamma}e_n = e_{n-1}$, $(U_t)_{\gamma}e_n = e^{i(n+\gamma)t}e_n$.

Proof. Let $(E, T, V, (U_t))$ be an irreducible quadruple; it is easy to see that the triple $(E, T, V)$ is a multiple of the unique irreducible triple which we denote by $(E_0, T_0, V_0)$, so that we can write, for some Hilbert space $K$: $E = E_0 \otimes K$, $T = T_0 \otimes 1$, $V = V_0 \otimes 1$, define a one parameter unitary group $(U_t)_0$ in $E_0$ by $(U_t)_0 e_n = e^{int} e_n$ and another in $E$ by $\tilde{U}_t = (U_t)_0 \otimes 1$. We have

R. Rebollo (ed.), Stochastic Analysis and Mathematical Physics
\[ \tilde{U}_t T \tilde{U}_t^{-1} = T, \tilde{U}_t V \tilde{U}_t^{-1} = e^{-it}V. \]

Hence \( \tilde{U}_t^{-1}U_t \) commutes with \( T \) and \( V \). Since \( (T_0, V_0) \) is irreducible, \( \tilde{U}_t^{-1}U_t \) can be written as \( 1 \otimes W_t \); the relation \( U_t = (U_t)_0 \otimes W_t \) implies \( W_{s+t} = W_s W_t \), i.e., \( W \) is a one parameter unitary group in \( K \), irreducible since each operator in \( K \) which commutes with it will also commute with \( T, V \) and \( (U_t) \); hence \( W_t \) is of the form \( e^{it} \) and \( K = \mathbb{C} \). This proves that \( E = E_0, T = T_0, V = V_0, U_t = e^{it(U_t)_0} \).

2 A \( C^* \)-Algebra for Triples \((E, T, V)\)

Let us set \( S_0 = e^{-T_0} \), a bounded and self-adjoint operator in \( E_0 \) since \( T_0 \) is positive; let \( \tilde{A} \) be the sub-\( C^* \)-algebra of \( \mathcal{L}(E_0) \) generated by \( S_0 \) and \( V_0 \); let \( A \) be the closed self-adjoint ideal of \( \tilde{A} \) generated by \( S_0 \).

**Lemma 2.1.** The algebra \( A \) is the set \( \mathcal{C}(\mathcal{E}_t) \) of all compact operators in \( E_0 \).

**Proof.** The operator \( S_0 \) is compact since its spectrum is the set of the numbers \( e^{-|t|_1} \); hence \( A \) is included in \( \mathcal{C}(\mathcal{E}_t) \); but \( A \) is irreducible, and this implies that \( A = \mathcal{C}(\mathcal{E}_t) \) (cf. [1]). \( \square \)

**Corollary 2.2.** There is a bijective correspondence between triples \((E, T, V)\) and representations \( \pi \) of \( A : \pi(V_0) = V, \pi(S_0) = e^{-T} \).

**Proof.** Each triple is a multiple of \((E_0, T_0, V_0)\) and each representation of \( A \) is a multiple of its natural representation in \( E_0 \). \( \square \)

3 A \( C^* \)-Algebra for Quadruples \((E, T, V, (U_t))\)

We have an action of the group \( \mathbb{R} \) by automorphisms of \( A : \alpha_t(a) = (U_t)_0 a((U_t)_0)^{-1} \); hence giving a quadruple \((E, T, V, (U_t))\) is equivalent to giving a pair \((\pi, \rho)\) where \( \pi \) is a representation of \( A \) and \( \rho \) a unitary representation in the same space, such that

\[ \pi(\alpha_t(a)) = \rho(t)\pi(a)\rho(t)^{-1}; \]

such a pair is called a *covariant representation* of the pair \((A, \mathbb{R})\).

One can construct a \( C^* \)-algebra, called the *crossed product* \( A \times_{\alpha} \mathbb{R} \), whose representations are in a one-one correspondence with the pairs \((\pi, \rho)\). Firstly, one defines a multiplication in the space \( L^1(\mathbb{R}, A) \):

\[ (\Phi_1 \Phi_2)(t) = \int \Phi_1(s)\alpha_s(\Phi_2(t-s))ds \]