The goal of this discussion is to explain how quantum groups arise in three-dimensional topological quantum field theories (TQFTs). Of course, “explain how” is not the job of science, and perhaps you will find other explanations more satisfying. Let me explain!

What is a three-dimensional TQFT? At the very least it gives a topological invariant of three-dimensional manifolds. That is, to each 3-manifold $X$ it assigns a complex number $Z_X$ and if the invariants for $X$ and $X'$ are different ($Z_X \neq Z_{X'}$), then $X$ and $X'$ are not diffeomorphic. A three-dimensional TQFT also gives invariants of knots and links. For example, if $K$ is a knot in ordinary 3-space, then we get a set of numerical invariants $\{I_K(\alpha)\}$ indexed by some finite set. The Jones polynomial of a knot, and similar polynomial invariants of knots, fit into this picture. As a historical note, Vaughan Jones [1] introduced his polynomial invariant in the mid-80s before the advent to topological quantum field theories. Those were introduced by Edward Witten in 1987 (following a suggestion of Michael Atiyah), first in 4 dimensions to give a quantum field-theoretic interpretation of Donaldson’s invariants of 4-manifolds. A few years later [2] he introduced a three-dimensional TQFT which reproduces the Jones polynomial and which is our concern here. The classical action of this field theory is the Chern-Simons invariant, which was introduced into geometry in the early 1970s. As a mathematician I must immediately point out that Witten’s methods, involving the path integral, are far from an established part of rigorous 1990s mathematics.

Shortly after Witten’s paper on quantum Chern-Simons invariants, Reshetikhin and Turaev [3] showed how to start with extremely complicated algebraic data—called a quantum group—and again produce the Jones polynomial and its generalizations. (Their work is completely rigorous.) Subsequently, they showed [4] how to use the same data to construct invariants of 3-manifolds. The construction of a complete TQFT from this algebraic data, which involves more than invariants of 3-manifolds and knots, has been folklore ever since, and now has been described completely [5].
remark here that instead of starting with a quantum group, one can start with certain "categorical" data instead.

The algebraic data of either a quantum group or its categorical equivalent is extremely complicated! One could hardly guess in advance that such data can produce invariants of knots and 3-manifolds. Nor can one easily construct algebraic data satisfying the necessary hypotheses. By contrast the classical Chern-Simons action is beautiful and simple! It is relatively easy to write down. One sees from the beginning that Lie groups enter the picture in a fundamental way. And if you are willing to accept the path integral (you shouldn’t!), then you have a nice geometric construction of the Jones polynomial and related 3-manifold invariants. This leads us to pose the following.

**Problem.** Start with the Chern-Simons action and construct the quantum group which gives the same 3-manifold and knot invariants.

The goal is to explain how to do this in a simple case. As noted, the Chern-Simons theory starts with a compact Lie group $G$ (and a piece of topological data which will be explained later). The Jones polynomial concerns the case $G = \text{SU}(n)$ for variable $n$. In the simple case we treat, $G$ is a *finite* group. This was first considered by Dijkgraaf and Witten [6]. The major simplification here is that the path integral is a finite sum, rather than an integral over an infinite-dimensional space, so is rigorously defined. So we immediately get a 3-manifold invariant, though it is rather simple and relatively uninteresting. The knot invariants are possibly more interesting; I don’t believe that they have been investigated fully. In any case our interest is in the quantum group and our strategy is this: We exploit the fact that the path integral is well defined to introduce generalizations of the path integral. Thus one ingredient in a three-dimensional TQFT is a "quantum Hilbert space" $E(Y)$ for every surface $Y$. In usual quantum field theories it is constructed by canonical quantization. In our simple model we show how to get it by an exotic path integral. Something is immediately very strange—the result of an integration is a Hilbert space! Even more strange is the path integral we introduce for a 1-manifold, i.e., for a circle $S$. There is where we will see the quantum group emerging. In fact, the quantum groups we compute this way were written down in a paper of Dijkgraaf et al. [7]. They did not related it to the Chern-Simons invariant. It was a conjecture of Altschuler and Coste [8] that these quantum groups construct (via the Reshetikhin-Turaev prescription) the invariants of the finite group Chern-Simons theory. Our methods prove this conjecture.

This, then, is our strategy. In a $d$-dimensional field theory, where usually the classical action is only defined for fields in $d$ dimensions, we will generalize the classical action to fields on manifolds of dimension less than $d$. We then introduce a corresponding generalization of the path integral for these exotic classical actions. Of course, one is immediately led to ask whether our constructions can be generalized, at least heuristically, in Chern-Simons