A SURVEY OF DISCRETE TRACE FORMULAS

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Abstract. The goal is to survey work on Selberg's trace formula for discrete quotient spaces \( G/K \) both finite and infinite. Here \( G \) is often the general linear group \( GL(n, F) \) consisting of \( n \times n \) non-singular matrices with entries in some field \( F \), and \( K \) is some subgroup. Usually \( F \) is the finite field \( \mathbb{F}_q \) with \( q \) elements and \( n = 2 \).

We begin with the trace formula for finite abelian groups (i.e., Poisson's summation formula) and an application to error-correcting codes. For non-abelian groups, we consider three main topics:

- an application of the pre-trace formula to find some isospectral non-isomorphic Schreier graphs with vertex sets \( GL(3, \mathbb{F}_2)/\Gamma_i, i = 1, 2 \), with \( \Gamma_1 \) consisting of matrices having first column equal to \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( \Gamma_2 \) the transpose of \( \Gamma_1 \);
- the trace formula for \( GL(2, \mathbb{F}_q)/GL(2, \mathbb{F}_p) \), where \( q = p^r \);
- the trace formula on the \( k \)-regular tree (which is a \( p \)-adic quotient space if \( k = p + 1 \)) and Ihara's theorem for the zeta function of a finite \( k \)-regular graph.

1. Introduction. Finite abelian trace formulas.

- T. Tamagawa [59]: “Selberg’s theory is one of the biggest guns.”
- D. Hejhal [26]: “Some readers may feel that Hejhal has created a 'monster'.”
- R. Langlands (in [4], p. 152): “All of this looks to be elaborate and extremely difficult, as indeed it is; so it is very helpful to understand it for simple examples ....”

Selberg’s trace formula for continuous Riemannian symmetric spaces \( G/K \) was formulated by Selberg [49] 40 years ago. Like the Poisson summation formula for abelian groups, the trace formula is closely related to the method of images in mathematical physics.

And, despite the slightly fearsome quotes above, the trace formula has proved to be an extremely useful tool. Applications include: Weyl’s law on the distribution of eigenvalues of the Laplacian on a Riemann surface, the work of Marie-France Vignéras [67] on Riemann surfaces that are isospectral but not isomorphic, results on the behavior of Selberg’s zeta function. For example, the analytic properties of the Selberg zeta function are essentially equivalent to the trace formula (see Elstrodt [13] and Sarnak [48]). We will summarize some of this in Section 3. Our main goal here is to consider discrete analogues of some of these results.

The simplest sort of trace formula says that the trace of an \( n \times n \) matrix of real or complex numbers is the sum of the diagonal entries as well as the sum of the eigenvalues of the matrix. For a finite abelian group \( G \)
order $n$, you can view a matrix $k = (k_{ij})_{1 \leq i, j \leq n}$ as a kernel $k(i, j), i, j \in G$. Then we are saying

$$\sum_{i \in G} k(i, i) = \sum_{i=1}^{n} \lambda_i,$$

where $\lambda_i$ is the $i$th eigenvalue of the matrix $k$. In order to stay close to the spirit of Selberg’s original trace formula, we wish to use the language of functional analysis and integral operators acting on the $n$-dimensional vector space $L^2(G)$ of complex-valued functions on $G$. So we view our matrix $k$ as an operator $L_k$ acting on functions $f : G \to \mathbb{C}$ via:

$$L_k f(i) = \sum_{j \in G} k(i, j) f(j), \quad \text{for } j \in G. \quad (2)$$

We will normally be considering an operator $L_k$ which is self-adjoint with respect to the inner product of functions $f, g \in L^2(G)$ given by

$$\langle f, g \rangle = \sum_{j \in G} f(j) \overline{g(j)}. \quad (I, g) = \langle L_k f, g \rangle \quad \text{for } i \in G.$$  

That is, usually we assume $\langle L_k f, g \rangle = \langle f, L_k g \rangle$; i.e., the matrix $k$ satisfies

$$k_{ij} = \overline{k_{ji}}. \quad (4)$$

If we assume that $L_k$ commutes with all the shift operators $T_g$ given by

$$T_g f(x) = f(x + g), \quad \text{for } x, g \in G. \quad (3)$$

then $L_k$ is actually convolution by a function. This follows from the existence of a function $h$ on $G$ such that

$$k(i, j) = h(i - j). \quad (4)$$

That is, if $k$ is as in (4), the operator $L_k = L_h$ in (2) is the convolution operator:

$$L_k f(i) = L_h f(i) = (f * h)(i) = \sum_{j \in G} h(i - j) f(j). \quad (5)$$

Let $G = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of integers modulo $n$ under addition. That is, we identify two integers $a, b$ if $n$ divides $a - b$. So we can think of $\mathbb{Z}/n\mathbb{Z}$ as a finite circle of $n$ points. This is the group on which the finite and fast Fourier transforms live. For this $G$, the matrix with entries $k(i, j)$