Differential equations have played an important part in the development of mathematics since the time of Newton. They have also been central to many applications of mathematics to the physical sciences and technology. Often the equations relevant to practical applications are so difficult to solve explicitly that they can only be handled with approximation techniques on large computer systems. In this chapter we will be concerned with a simple form of differential equation, and systems thereof, namely, linear differential equations with constant coefficients. These systems have a great deal in common with systems of linear equations, and we are in a position to apply the hard won knowledge about Jordan canonical forms to solve such systems.

18.1 Linear Differential Systems: Basic Definitions

Differential equations come in all sizes. Here is a relatively simple example.

**EXAMPLE 1:** Find all differentiable functions $y : \mathbb{R} \rightarrow \mathbb{R}$ with the property that the derivative $d y / dt$ of $y$ satisfies the equation

$$
\frac{dy}{dt}(t) = r \cdot y(t), \quad \forall \ t \in \mathbb{R},
$$

where $r$ is some fixed number.

**SOLUTION:** $y = ce^{rt}$, where $c$ is a constant, as follows from the fundamental theorem\(^1\) of the calculus.

We begin by defining some basic concepts.

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\(^1\) By the **fundamental theorem of the calculus** we understand the following
**Definition:** Let \( n \) be a positive integer, \( B \subseteq \mathbb{R}^{n+2} \) a nonempty subset, and
\[
F : B \rightarrow \mathbb{R}
\]
a function. A function
\[
y : J \rightarrow \mathbb{R}
\]
defined on the interval \( J = [a, b] \subseteq \mathbb{R} \) (i.e., \( J = \{ t \in \mathbb{R}, a \leq t \leq b \} \)) is called a solution to the differential equation
\[
F\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^n y}{dt^n}\right) = 0
\]
on the interval \( J \) if the following conditions are fulfilled:
1. \( y \) is an \( n \)-times differentiable function on \( J \).
2. \( \forall t \in J \) the \((n+2)\)-tuple
\[
(t, y(t), \frac{dy}{dt}(t), \ldots, \frac{d^n y}{dt^n}(t))
\]
belongs to \( B \).
3. \( \forall t \in J \) we have
\[
F\left(t, y(t), \frac{dy}{dt}(t), \ldots, \frac{d^n y}{dt^n}(t)\right) = 0.
\]

A class of differential equations of particular interest are those of the form
\[
\frac{d^n y}{dt^n} - f\left(t, y, \ldots, \frac{d^{n-1} y}{dt^{n-1}}\right) = 0,
\]
or, equivalently,
\[
\frac{d^n y}{dt^n} = f\left(t, y, \ldots, \frac{d^{n-1} y}{dt^{n-1}}\right).
\]
Such an equation is said to be of \( n \)-th order.

We will also consider systems of differential equations. They are analogous to systems of simultaneous equations.

**Theorem:**

**Theorem 18.1.1** (Fundamental Theorem of the Calculus): If \( f(t) \) is a continuous function on the interval \([a, b]\) and
\[
F(x) = \int_a^x f(t) dt \quad \forall x \in [a, b]
\]
then \( F'(x) = f(x) \) for all \( x \in (a, b) \). If \( H(x) \) is any other function with \( H'(x) = f(x) \forall x \in (a, b) \) then \( H(x) = F(x) + C \) for some constant \( C \in \mathbb{R} \).