COVARIANCE MATRIX COMPUTATION OF THE STATE VARIABLE OF A STATIONARY GAUSSIAN PROCESS

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1. Summary

A recursive procedure for the computation of one-step ahead predictions for a finite span of time series data by a Gaussian autoregressive moving average model can be realized by using the Markovian representation of the model. The covariance matrix of the stationary state variable of the Markovian representation is required to implement a computational procedure of the predictions. A simple computational procedure of the covariance matrix which does not need an iterative method is obtained by using a canonical representation of the autoregressive moving average process. The recursive computation of the predictions realized by using this procedure provides a computationally efficient method of exact likelihood evaluation of a Gaussian autoregressive moving average model.

2. Recursive computation of one-step ahead predictions

Assume that a stationary scalar zero-mean Gaussian process $y(n)$ is defined by the relation

$$
    z(n+1) = Fz(n) + Gx(n+1), \quad y(n) = Hz(n)
$$

where $x(n)$ is a scalar white noise independent of $z(n-1), z(n-2), \ldots$, and $z(n)$ is the state variable which is a stationary $p$-vector process and the matrices $F$, $G$, $H$ are $p \times p$, $p \times 1$ and $1 \times p$, respectively. When $y(0), y(1), \ldots, y(n-1)$ are given, the one-step ahead prediction of $z(n)$ is defined by

$$
    z(n|0, n-1) = \text{projection of } z(n) \text{ onto the linear space spanned by the components of } y(0), y(1), \ldots, y(n-1),
$$

and the one-step ahead prediction of $y(n)$ is given by $y(n|0, n-1) = Hz(n|0, n-1)$.

The computation of $z(n|0, n-1)$ can be done recursively by using...
the relations
\[ e(n) = y(n) - Hz(n|0, n-1) , \]
(2.2)
\[ z(n+1|0, n) = Fz(n|0, n-1) + K_n r_n^{-1} e(n) , \]
\[ z(0|0, -1) = 0 , \]
where \( r_n = E e(n)^2 \) and \( K_n = E z(n+1|0)e(n) \), the Kalman gain vector. By the Kalman filtering procedure, which is a standard procedure of recursive computation of the predictions, \( K_n \) and \( R_n \) are computed by the relations
(2.3)
\[ K_n = FP(n|n-1)H', \quad r_n = HP(n|n-1)H' , \]
where \( ' \) denotes transpose and \( P(n|n-1) = E(z(n) - z(n|0, n-1))(z(n) - z(n|0, n-1))' \) and is obtained by the relations
(2.4)
\[ P(n+1|n) = FP(n|n-1)F' + GqG' - K_n r_n^{-1} K_n' , \]
\[ P(0|-1) = P_0 , \]
where \( q = E x(n)^2 \) and \( P_0 = E z(0)z(0)' \), the covariance matrix of the stationary state vector. Thus the numerical evaluation of \( P_0 \) forms the starting point of the Kalman filtering procedure.

3. Prediction of ARMA process

When \( y(n) \) is a stationary Gaussian autoregressive moving average (ARMA) process defined by
(3.1)
\[ y(n) + b_1 y(n-1) + \cdots + b_M y(n-M) = x(n) + a_1 x(n-1) + \cdots + a_L x(n-L) \]
there is a Markovian representation (2.1) defined by
\[ F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_K & -b_{K-1} & -b_{K-2} & \cdots & -b_1 \end{bmatrix} , \quad G = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{K-2} \\ w_{K-1} \end{bmatrix} , \]
\[ H = [1 \ 0 \ 0 \ \cdots \ 0] , \]
where \( a_L \neq 0 \), \( b_M \neq 0 \) and \( K = \max (M, L+1) \) and the \( w_i \)'s are the impulse responses of the system (3.1), i.e., \( (w_0, w_1, \cdots, w_{K-1}) = (y(0), y(1), \cdots, y(K-1)) \) under the assumption that \( y(-1) = y(-2) = \cdots = y(-M) = 0 \) and \( x(n) = 1 \), for \( n = 0 \), 0, otherwise. The corresponding state variable \( z(n) \)