Curves are the most basic geometric objects. In this chapter, we will study curves in the plane and curves in three-dimensional space. To each curve, we can attach certain natural geometric invariants; that is, quantities that express physical properties of the curve. These invariants include:

- arclength;
- the number of singularities (such as cusps);
- curvature, which measures how much the curve bends; and
- torsion, which measures how much the curve twists.

We will use calculus to define each invariant in terms of derivatives and integrals. Finally, we will show that the geometric invariants characterize the curve. In other words, distinct curves cannot have identical invariants, unless they are congruent.

**Parametric Curves**

A curve is the image of a continuous (and usually differentiable) function \( r : I \to \mathbb{R}^n \), where \( I \) is an interval. In this book, \( n \) is either 2 or 3. We shall be ambiguous about the nature of the interval \( I \), allowing for the possibility of it being open, closed, bounded, or unbounded. Here are some examples of curves.

- The right circular helix of radius 1:
  \[
  r(t) = (\cos t, \sin t, t), \quad t \in \mathbb{R}.
  \]
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- A three-dimensional astroid:
  \[ r(t) = (\cos^3 t, \sin^3 t, \cos 2t), \quad 0 \leq t \leq 2\pi. \]

- The cycloid is the curve traced out by a point on the circumference of a wheel as it rolls in a straight line at constant speed without slipping. The parametric equations are:
  \[ r(t) = (t - \sin t, 1 - \cos t), \quad t \in \mathbb{R}. \]

The cycloid is a plane curve; the helix and astroid are space curves. Throughout the first part of this chapter, we will use the helix as our illustrative example. We will return to the other examples later in the chapter.

We begin by defining the helix in Mathematica and then graphing it.

\[
\text{In}[1]:= \text{helix} = \{\text{Cos}[t], \text{Sin}[t], t\}
\]

\[
\text{Out}[1]= \{\text{Cos}[t], \text{Sin}[t], t\}
\]

\[
\text{In}[2]:= \text{helPlot} = \text{ParametricPlot3D}[\text{Evaluate[helix]}, \{t, -2\pi, 2\pi\}, \text{ViewPoint} \rightarrow \{1, 1, 1\}, \text{Ticks} \rightarrow \text{None}];
\]

The function \( r(t), t \in I \), is called a parametrization of the curve; the curve itself is the physical set of points (in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)) traced out by the function \( r \) as \( t \) varies in the interval \( I \). Every curve has many different possible parametrizations. All of the geometric invariants we associate to a curve will be computed analytically in terms of a parametrization. Clearly, if there is to be any geometric validity for an invariant, the quantity should be independent of the choice of parametrization.