Chapter 9. Excess and Residual Intersections

Summary

If $X \subset Y$ is a regular imbedding, $V \subset Y$ a subvariety, we have constructed (§ 6.1) an intersection product $X \cdot V$ in $A_m(X \cap V)$, where $m = \dim V - \text{codim}(X, Y)$. If a closed subscheme $Z$ of $X \cap V$ is given, the basic problem of residual intersections is to write $X \cdot V$ as the sum of a class on $Z$ and a class on a "residual set" $R$. There is a canonical choice for the class on $Z$, namely

$$\{c(N) \cap s(Z, V)\}_m$$

where $N$ is the restriction to $Z$ of $N_X Y$, and $s(Z, V)$ is the Segre class. Our problem is therefore to compute this class on $Z$, and to construct and compute a residual intersection class $R$ in $A_m(R)$, for an appropriate closed set $R$ such that $Z \cup R = X \cap V$, with

$$X \cdot V = \{c(N) \cap s(Z, V)\}_m + R.$$  

If $m = 0$, and $R$ is a finite set, knowing $X \cdot V$ and $\{c(N) \cap s(Z, V)\}_0$ gives a formula for the weighted number of points of $R$. This is the basis for applications of the excess intersection formula to enumerative geometry.

In case $Z$ is a (scheme-theoretic) connected component of $X \cap V$, $R$ is the union of the other connected components; since $A_*(X \cap V) = A_*(Z) \oplus A_*(R)$, the above decomposition is part of the construction of Chap. 6. Computations, applications, and a few of the many classical examples are considered in § 9.1.

The general case is considered in § 9.2. In the main theorem $Z$ is assumed to be a Cartier divisor on $V$; in this case there is a natural scheme structure on the residual set, which can be used to construct $R$. If $Z$ is arbitrary, one blows up $V$ along $Z$ to reduce to the divisor case.

An important and typical application of the residual intersection theorem is to the formula for the double point cycle class of a morphism, which is given in § 9.3.

9.1 Equivalence of a Connected Component

Let $Y$ be a scheme, $X_i \subset Y$ regularly imbedded subschemes, $1 \leq i \leq r$, and $V$ a $k$-dimensional subvariety of $Y$. The intersection product

$$X_1 \cdot \ldots \cdot X_r \cdot V$$

W. Fulton, Intersection Theory
Chapter 9. Excess and Residual Intersections

is a class in $A_m(\bigcap X_i \cap V)$, $m = \dim V - \sum_{i=1}^r \text{codim}(X_i, Y)$. It is constructed by the procedure of §6.1, applied to the diagram

\[
\begin{array}{c}
\bigcap X_i \cap V \\
\hookrightarrow V \\
X_1 \times \ldots \times X_r \hookrightarrow Y \times \ldots \times Y.
\end{array}
\]

If $Z$ is a connected component of $\bigcap X_i \cap V$, we write

\[ (X_1, \ldots, X_r \cdot V)^Z \in A_m(Z) \]

for the part of $X_1, \ldots, X_r \cdot V$ supported on $Z$, and call it the equivalence of $Z$ for the intersection $X_1, \ldots, X_r \cdot V$.

**Proposition 9.1.1** Let $N_i$ be the restriction of $N_{X_i, Y}$ to $Z$. Then

\[ (X_1, \ldots, X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_i) \cap s(Z, V) \right\}_m. \]

If $Z$ is regularly imbedded in $V$, with normal bundle $N_{Z, V}$, then

\[ (X_1, \ldots, X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_i) \cdot c(N_{Z, V})^{-1} \cap [Z] \right\}_m. \]

If $V$ and $Z$ are non-singular, then

\[ (X_1, \ldots, X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_i) \cdot c(T_{V|Z})^{-1} \cap [Z] \right\}_m. \]

**Proof.** The first assertion follows from Proposition 6.1(a) and the Whitney sum formula (Theorem 3.2(e)). The second follows from Proposition 4.1(a). The last uses the identification of $N_{Z, V}$ with the quotient of tangent bundles $T_{V|Z}/T_Z$ (Appendix B.7.2) and the Whitney sum formula. □

The global intersection class $X_1, \ldots, X_r \cdot V$ is always the sum of the equivalences for the connected components of $\bigcap X_i \cap V$. The following case suffices for many enumerative applications.

**Proposition 9.1.2.** Suppose $m = 0$, and $\bigcap X_i \cap V$ consists of a connected component $Z$ together with a finite set $S$. Then the degree of $X_1, \ldots, X_r \cdot V$ is

\[ \text{deg}(X_1, \ldots, X_r \cdot V)^Z + \sum_{P \in S} i(P, X_1, \ldots, X_r \cdot V; Y) \cdot [R(P) : K], \]

where $[R(P) : K]$ denotes the degree of the residue field of $P$ over the ground field. □

In enumerative applications, $\text{deg}(X_1, \ldots, X_r \cdot V)$ is known for global reasons, e.g. Bézout's theorem. Knowing the equivalence of $Z$ then predicts the (weighted) number of residual points. Following the classical terminology, when $m = 0$, we also call the degree of $(X_1, \ldots, X_r \cdot V)^Z$ the equivalence of $Z$ in the intersection product. If $V = Y$, we write $(X_1, \ldots, X_r)^Z$ in place of $(X_1, \ldots, X_r \cdot V)^Z$. 