Chapter 5. Deformation to the Normal Cone

Summary

If $X$ is a closed subscheme of $Y$, there is a family of imbeddings $X \hookrightarrow Y_t$, parametrized by $t \in \mathbb{P}^1$, such that for $t = 0$ (in fact for $t \neq \infty$) the imbedding is the given imbedding of $X$ in $Y$, and for $t = \infty$ one has the zero section imbedding of $X$ in the normal cone $C_X Y$. The existence of such a deformation, together with the "principle of continuity" that intersection products should vary nicely in families, explains the prominent role to be played by the normal cone in constructing intersection products.

5.1 The Deformation

Let $X$ be a closed subscheme of a scheme $Y$, and let $C = C_Y X$ be the normal cone to $X$ in $Y$. We will construct a scheme $M = M_x Y$, together with a closed imbedding of $X \times \mathbb{P}^1$ in $M$, and a flat morphism $q : M \to \mathbb{P}^1$ so that

\[
\begin{array}{ccc}
X \times \mathbb{P}^1 & \hookrightarrow & M \\
pr & \downarrow & q \\
\mathbb{P}^1 & & \emptyset
\end{array}
\]

commutes, and such that:

(1) Over $\mathbb{P}^1 - \{ \infty \} = \mathbb{A}^1$, $q^{-1}(\mathbb{A}^1) = Y \times \mathbb{A}^1$ and the imbedding is the trivial imbedding:

\[
X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1.
\]

(2) Over $\infty$, the divisor $M_\infty = q^{-1}(\infty)$ is the sum of two effective Cartier divisors:

\[
M_\infty = P(C \oplus 1) + \tilde{Y}
\]

where $\tilde{Y}$ is the blow-up of $Y$ along $X$. The imbedding of $X = X \times \{ \infty \}$ in $M_\infty$ is the zero-section imbedding of $X$ in $C$, followed by the canonical open imbedding of $C$ in $P(C \oplus 1)$. The divisors $P(C \oplus 1)$ and $\tilde{Y}$ intersect in the scheme $P(C)$, which is imbedded as the hyperplane at infinity in $P(C \oplus 1)$, and as the exceptional divisor in $\tilde{Y}$.
In particular, the image of $X$ in $M_\infty$ is disjoint from $\bar{Y}$. Letting $M^\circ = M_\infty \setminus \bar{Y}$, one has a family of imbeddings of $X$:

$$X \times \mathbb{P}^1 \hookrightarrow M^\circ \hookrightarrow \mathbb{P}^1$$

which deforms the given imbedding of $X$ in $Y$ to the zero-section imbedding of $X$ in $C_X Y$.

To construct this deformation, let $M$ be the blow-up of $Y \times \mathbb{P}^1$ along the subscheme $X \times \{\infty\}$. Since the normal cone to $X \times \{\infty\}$ in $Y \times \mathbb{P}^1$ is $C \oplus 1$, the exceptional divisor in this blow-up is $P(C \oplus 1)$.

From the sequence of imbeddings

$$X = X \times \{\infty\} \hookrightarrow X \times \mathbb{P}^1 \hookrightarrow Y \times \mathbb{P}^1,$$

the blow-up of $X \times \mathbb{P}^1$ along $X \times \{\infty\}$ is imbedded as a closed subscheme of $M$ (Appendix B.6.9); since $X \times \{\infty\}$ is a Cartier divisor on $X \times \mathbb{P}^1$, the blow-up of $X \times \mathbb{P}^1$ along $X \times \{\infty\}$ may be identified with $X \times \mathbb{P}^1$, so we have a closed imbedding

$$X \times \mathbb{P}^1 \hookrightarrow M.$$

Similarly from

$$X \times \{\infty\} \hookrightarrow Y \times \{\infty\} \hookrightarrow Y \times \mathbb{P}^1$$

the blow-up $\bar{Y}$ of $Y$ along $X$ is imbedded as a closed subscheme of $M$.

Since the projection from $Y \times \mathbb{P}^1$ to $\mathbb{P}^1$ is flat, the composite

$$\varrho : M \to \mathbb{P}^1$$

of the blow-down map from $M$ to $Y \times \mathbb{P}^1$ followed by the projection to $\mathbb{P}^1$ is flat (Appendix B.6.7).

Since $M \to Y \times \mathbb{P}^1$ is an isomorphism away from $Y \times \{\infty\}$, assertion (1) is clear. The description (2) of $M_\infty = \varrho^{-1}(\infty)$ as the sum of Cartier divisors will follow from the explicit algebraic description given below; since we have $P(C \oplus 1)$ and $\bar{Y}$ globally imbedded in $M$, it suffices to examine their structure locally on $Y$.

Assume $Y = \text{Spec}(A)$, and $X$ is defined by the ideal $I$ in $A$. To study $M$ near $\infty$, identify $\mathbb{P}^1 \setminus \{0\}$ with $\mathbb{A}^1 = \text{Spec}K[T]$, where $K$ is the ground field. The blow-up of $Y \times \mathbb{A}^1$ along $X \times \{0\}$ is $\text{Proj}(S')$, with

$$S^n = (I, T)^n = I^n + I^{n-1}T + \ldots + AT^n + AT^{n+1} + \ldots.$$

$\text{Proj}(S')$ is covered by affine open sets $\text{Spec}(S'_a)$, where $S'_a$ is the ring of fractions

$$S'_a = \{s/a^n \mid s \in S^n\},$$

and $a$ runs through a set of generators for the ideal $(I, T)$ in $A[T]$. For $a \in I$, the exceptional divisor $P(C \oplus 1)$ is defined in $\text{Spec}(S'_a)$ by the equation $a/1, a \in S^0$, while $\bar{Y}$ is defined by $T/a$; since $T = (a/1) \cdot (T/a)$, the description of $M_\infty$ as the sum of $P(C \oplus 1)$ and $\bar{Y}$ follows.

The complement of $\bar{Y}$ in the blow-up of $Y \times \mathbb{A}^1$ along $X \times \{0\}$ is $\text{Spec}(S_{(T)})$, where

$$S_{(T)} \cong \ldots \oplus I^n T^{-n} \oplus \ldots \oplus I T^{-1} \oplus A \oplus AT \oplus \ldots \oplus AT^n \oplus \ldots.$$