CHAPTER 3

Groupoid $C^*$-Algebras and Their Relation to Inverse Semigroup Covariance $C^*$-Algebras

3.1 Representation theory for locally compact groupoids

Let $G$ be a locally compact groupoid. We recall (Definition 2.2.2) that $G$ is equipped with a left Haar system $\{\lambda^u\}$.

In the representation theory of locally compact groups, unitary representations of such a group $G$ are integrated up with respect to left Haar measure against $C_c(G)$ (or even $L^1(G)$) functions to give a representation of the convolution algebra $C_c(G)$. In the locally compact groupoid case, however, there are many measures involved in the Haar system. Each unit $u$ has “its own” measure, $\lambda^u$, which lives on the “little piece” $G^u = r^{-1}(\{u\})$ of the groupoid. Leaving aside exactly what a representation $L$ of the groupoid $G$ should be for the moment and thinking very roughly, for any $f \in C_c(G)$ and $u \in G^0$, we will have to integrate up $x \mapsto f(x)L(x)$ in some sense over $G^u$ with respect to $\lambda^u$ and then combine together what we get over $u \in G^0$.

The natural mathematical framework for this is that in which we have a (probability) measure $\mu$ and a Hilbert bundle $\{H_u\} (u \in G^0)$. Each $L(x)$ will then be an operator from $H_{d(x)}$ to $H_{r(x)}$ and the above “combining”
will be done by integrating with respect to $\mu$.\footnote{Strictly, a “modular function” needs also to be taken into account.}

The requirement that groupoid representation theory should involve a probability measure $\mu$ on the unit space $G^0$ seems initially unfamiliar from a group point of view, since representation theory for a locally compact group does not explicitly use such a measure. However, in that case, there is such a measure implicitly involved – the point mass at the single unit of the group, the identity! Of course in that context, it is not worth the trouble to mention it! By contrast, a groupoid with a large unit space will have many measures $\mu$ relevant for representation theory.

However, as one would expect, not every probability measure on $G^0$ will be relevant for representation theory. By considering the representation theory of transformation groupoids ([179, Ch.1])), such a measure should satisfy some kind of invariance condition. The condition that we need is that of quasi-invariance. We now describe in more detail representation theory for locally compact groupoids. This description is based on Renault’s account in [230].\footnote{In the approaches of Renault ([230]) and Muhly ([179]), results of Bourbaki ([19]) and Effros ([82]) respectively are used to facilitate the measure theory. However, in the present (non-Hausdorff) context, it seemed preferable to work out the details from scratch. This also gives a self-contained account, and indeed, as we will see, the details, while requiring care, use only basic measure theory.}

If $X$ is locally compact Hausdorff, then the set of probability measures on $X$ is denoted by $P(X)$. Let $G$ be a locally compact groupoid. Since $G$ (and $G^2$) need not be Hausdorff, a little care is needed in places with the measure theory.

A positive Borel measure on $G$ is a $[0, \infty]$-valued measure on the Borel $\sigma$-algebra $\mathcal{B}(G)$. Regularity for a positive Borel measure on $G$ is defined exactly as for the locally compact Hausdorff case ([120, p.127]). (There are many compact (Hausdorff) subsets of $G$ available for the purposes of inner regularity by (ii) of Definition 2.2.1.)

Let $\mu \in P(G^0)$. Then $\mu$ and the left Haar system determine a regular Borel positive measure $\nu$ on $\mathcal{B}(G)$ conveniently written:

$$\nu = \int_{G^0} \lambda^u \, d\mu(u).$$

More precisely, let $U$ be an open subset of $G$ contained in some open Hausdorff subset $U'$ of $G$ such that the closure of $U$ in $U'$ is compact. By Urysohn’s lemma, there exists $F \in C_c(G)$ such that $F \geq \chi_U$. Define a linear functional $\phi_U$ on $C_c(U)$ by defining

$$\phi_U(f) = \int d\mu(u) \int f(x) \, d\lambda^u(x).$$