**Error Bounds for a Stiff Markov Chain Approximation Technique and an Application**

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**Abstract:** A classical stiff Markov chain solution technique is adapted to analysis of dependability models, and given a new interpretation. This allows the derivation of bounds for the approximation error. A numerical example illustrates the practical use of these error bounds.

**Keywords and phrases:** Markov Chain, dependability, transient solution, approximation, error bounds

### 13.1 Introduction

Continuous-time Markov chains (CTMC) are widely used models in the fields of dependability and performance evaluation. The problem of computing the steady-state probability vector has been thoroughly studied, but few methods are available for the transient analysis of a Markov model [Stewart (1994)].

The computation of the transient probabilities is especially difficult for large and *stiff* models. The most commonly accepted definition of stiffness is the presence of “fast states”, i.e., states with average sojourn time much smaller than observation time $t$ [Dunkel and Stahl (1993), Malhotra et al. (1994)].

Several authors have suggested techniques to reduce the size of the graph, and to avoid (or tolerate) stiffness. In 1986, Bobbio and Trivedi [Bobbio and Trivedi (1986)] proposed an approximation technique based on a classification of the state space, generalizing previous work on the subject [Courtois and Semal (1984), McGough et al. (1985), White (1991)]. The method has been further analyzed and developed in [Bobbio and Trivedi (1990)] and [Reibman et al. (1990)].
We have adapted and used this method to analyze stiff dependability models. Although the results seem generally accurate, the general validity of the approximation had to be demonstrated through the derivation of error bounds. We indeed show with a counter-example that the computation of a system unavailability using this method may not be conservative; yet possibly underestimated risk values are not acceptable in the field of dependability.

To our knowledge the problem of bounding the approximation error has not yet been solved, except in the more specific context of "instantaneous coverage approximation" [White (1991)].

The paper is organized as follows. Section 13.2 briefly introduces the main notations used in the article. In Section 13.3, we suggest a "path-based" method to compute the state vector. In Section 13.4, this method is connected to Bobbio and Trivedi's algorithm. This enables us to bound the approximation error of this classical algorithm (Theorem 13.4.1). Section 13.5 presents a numerical example. The appendix contains the proofs of our main results.

### 13.2 Notations

We consider a continuous-time Markov chain \( \{X(t); t \geq 0\} \) on a finite state space \( \mathcal{E} \). Let \( A \) denote its transition rate matrix (or infinitesimal generator). Entry \( a_{ij} (i \neq j) \) of matrix \( A \) is the transition rate from \( i \) to \( j \). The exit rate of state \( i \) is defined as

\[
    a_i = \sum_{j \in \mathcal{E}; j \neq i} a_{ij},
\]

and the diagonal entry of \( A \) is \( a_{ii} = -a_i \).

The subject of this paper is the computation of the state vector \( P(t) \) at time \( t \), given its initial value \( P(0) \).

### 13.3 Approximation Techniques

#### 13.3.1 A path-based technique

It is well known that \( P(t) \) may be computed as the solution of the Chapman-Kolmogorov differential system, or from the exponential of matrix \( tA \). We will though focus our attention on another expression of \( P(t) \).

Given a state \( j \) in \( \mathcal{E} \), let us consider all the sequences \( s \) of states that may have been visited before the process hit \( j \) for the last time. These sequences are called "paths". We denote by \( \mathcal{P}_j \) the set of all these paths: