Asymptotic Results for the Failure Time of Consecutive $k$-out-of-$n$ Systems

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Abstract: We consider a consecutive $k$-out-of-$n$ system. This system fails if and only if there are at least $k$ consecutive failed components. Our goal is to state asymptotic results concerning the failure time $Z_n$ of the system. $Z_n$ is given by: $Z_n = \min_{1 \leq j \leq n-k+1} \max_{j \leq i \leq j+k-1} T_i$, where $T_1, \ldots, T_n$ denote the failure times of components. We establish strong laws of Erdos-Rényi-Shepp type for the statistics $Z_n$ and $k_n^{\frac{a}{\log k_n}}$ for $k_n = \lfloor c \log n \rfloor$ and a suitable constant $a$. We suppose that the components are independent and with equal failure distributions.

Keywords and phrases: Consecutive $k$-out-of-$n$ systems, failure time, strong laws

17.1 Introduction

A consecutive $k$-out-of-$n$ system consists of $n$ components disposed linearly. The system fails if and only if at least $k$ consecutive components are failed. Each component and the system has two states: it is functional or failed. This kind of systems has great importance in application. They have been proposed to model telecommunication systems and oil pipelines. Many papers gave methods to calculate reliability of such systems [see, for example, Derman, Lieberman and Ross (1982), Canfield and McCormick (1992)]. Some limit theorems are proved in [Papastavridis (1987), Chrissaphinou and Papastavridis (1990)]. In our paper we suppose that $k$ grows with $n$ (we note $k_n$) and we establish strong laws of Erdos-Rényi-Shepp type [Ksir (1989), Shepp (1964), Deheuvels, Devroye and Lynch (1986)] for the failure time of the system when the components are supposed independent and identically distributed.
### 17.2 Strong Laws for the Failure Time of the System

We suppose that the components are statically independent and with equal failure distributions. So, the failure times $T_1, ..., T_n$ of the components of the system are (positive) independent random variables identically distributed. The failure time of the system is: $Z_n = \min_{1 \leq j \leq n-k_n+1} \max_{i \leq j < i+j-k_n-1} T_i$. We are interested with the asymptotic behaviour of the random variables $Z_n$ and $k_n \frac{Z_n-a}{\log k_n}$ for the choice $k_n = \lceil c \log n \rceil$, $c > 0$, ($\lfloor x \rfloor$ represents the integer part of $x$), and a constant satisfying an equation given below.

Let us consider $m = E(T_1)$, $M = \text{ess-sup}T_1$, $h$ the Cramer transformation of $T_1$ and $t^*$ verifying: $h(a) = at^* - \log E(\exp(t^* T_1))$. We take $a \in [m, M]$ such that: $\exp(-h(a)) = \exp \left( -\frac{1}{c} \right)$. Under these considerations we state the following results:

**Theorem 17.2.1** $\lim_{n \to \infty} \sup Z_n \leq a$ a.s.

**Proof.**

\[
\Pr(Z_n > a) = \Pr\left( \min_{1 \leq j \leq n-k_n+1} \max_{i \leq j < i+j-k_n-1} T_i > a \right)
\]

\[
= \Pr\left( \min_{1 \leq j \leq n-k_n+1} Y_j > a \right)
\]

where $Y_j = \max_{j \leq i \leq j+k_n-1} T_i$. But $\min_{1 \leq j \leq n-k_n+1} Y_j \leq \frac{1}{n-k_n+1} \sum_{j=1}^{n-k_n+1} Y_j$, so:

\[
\Pr(Z_n > a) \leq \Pr\left( \frac{1}{n-k_n+1} \sum_{j=1}^{n-k_n+1} Y_j > a \right)
\]

\[
\leq \Pr\left( \sum_{j=1}^{n-k_n+1} Y_j > (n-k_n+1) a \right)
\]

\[
\leq \Pr\left( \sum_{j=1}^{n-k_n+1} Y_j - (n-k_n+1) a > 0 \right)
\]

\[
\leq E \left( \exp \left( t \left( \sum_{j=1}^{n-k_n+1} Y_j - (n-k_n+1) a \right) \right) \right), \forall t > 0
\]

\[
\leq E \left( \exp \left( t^* \left( \sum_{j=1}^{n-k_n+1} Y_j - (n-k_n+1) a \right) \right) \right) \text{ for } t = t^*
\]