Solutions to problems from part 7

19.1 Chapter 2

J. Zhang [1] has studied the system (2.19)–(2.22) which is a conservation law, with nonlocal terms, in the bounded region \( \{ x_1 > 0, x_2 > 0, \ldots, x_m > 0, x_1 + x_2 + \cdots + x_m = 1 \} \). He proved existence and uniqueness of the solution, and obtained numerical results for the intermediate and long term behavior of the solution when the initial data consists of several "hills."

19.2 Chapter 3

The time dependent problem of crack propagation in elastic medium was studied by A. Friedman and Y. Liu [2] in the 2-D case. The stress potential \( \varphi \) is a solution of \( \Delta^2 \varphi = 0 \) in the region outside the crack whereas, along the crack \( \Gamma_t = \{ X = X(s), 0 \leq s \leq t \} \),

\[
\varphi = 0, \quad \frac{\partial \varphi}{\partial n} = 0
\]

from both sides of \( \Gamma_t \). At the tip of the crack there holds:

\[
\dot{X}(t) = g(|\dot{X}(t)|) \, \overline{J}(X(t))
\] (19.1)

where the so called \( J \)-integral \( J(X(t)) \) is a nonlinear, nonlocal function in the second derivatives of \( \varphi \).

The law (19.1) is based upon:

(i) The principle that \( G = \gamma \) where \( G \) is the energy release rate at the crack’s tip and \( \frac{1}{2} \gamma \) is the stored fracture energy;

(ii) An experimental law which asserts that (for some materials) \( \gamma = g(|X|) \) where \( g \) is a known function which depends on the material properties, and

(iii) the formula \( G = \overline{J} (\overline{X}(t)) \cdot (\overline{X}(t)/|\dot{X}(t)|) \) plus a variational principle asserting that the tip \( X(t) \) chooses the direction \( \dot{X}(t)/|\dot{X}(t)| \) in such a way so as to maximize the amount of stress energy in the material that the crack can absorb as (surface) stored energy.
They considered in [2] a specific problem which is a perturbation of a travelling wave solution in an infinite strip \([-\infty < x < \infty, -a < y < a]\) with uniform crack propagating along the \(x\)-axis.

19.3 Chapter 4

A. Friedman and C. Huang [3] modeled the electrostatic paint spraying process as a continuous model of charged particles. Each particle, with initial position \(x\) and initial velocity \(v_\star\) then follows a trajectory \(\psi(x, t)\) subject to the law

\[
\frac{d^2\psi}{dt^2} + \lambda \frac{d\psi}{dt} = -\nabla \varphi(\psi, t) \quad (\psi(x, 0) = x, \; \psi_t(x, 0) = v_\star)
\]  

where \(\lambda \frac{d\psi}{dt}\) is the Stokes’ drag and \(-\nabla \varphi\) is the electric force. The electric potential \(\varphi\) satisfies the Poisson equation

\[
\Delta \varphi(x, t) = -P(x, t)
\]  

where \(P(x, t)\) is the distribution of mass (or charge), and the law of conservation of mass (or charge) is

\[
P(x, t) = P_0(\psi^{-1}(x, t))J(\psi^{-1})(x, t)
\]  

where \(J(\psi^{-1})\) is the Jacobian of the inverse mapping \(x \rightarrow \psi^{-1}(x, t)\), and \(P_0(x)\) is the initial distribution.

In [4] they studied the system (19.2)-(19.4) in the whole space \(\mathbb{R}^n\) and established local existence and uniqueness as well as global existence under some restrictions on the initial data (Global existence is not always valid.). In [3] they considered the system (19.2)-(19.4) in a confined geometry as in the spray painting model, with \(\varphi = M\) at the surface from which the paint particles originate and \(\varphi = 0\) on the workpiece. They have shown numerically that for a convex (or hill-shaped) part of the workpiece, the thickness of accumulated paint is smaller at the top than down along the slopes.

Introducing the Euler variables \(y = \psi(x, t)\) and \(v = \psi_t(x, t)\) the system (19.2)-(19.4) takes the more familiar form

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v + \lambda v = -\nabla \varphi(y, t),
\]

\[
\Delta \varphi(y, t) = -P(y, t),
\]

\[
\frac{\partial P}{\partial t} + \nabla(Pv) = 0,
\]

called the Euler-Poisson system. However, it is easier to work with the original formulation (19.2)-(19.4) since particles do leave the domain where the system is satisfied.