DECOMPOSITIONS OF D1 MODULES

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ABSTRACT. Continuity and quasicontinuity for modules may be viewed as generalizations of quasi-injectivity. A key property of quasi-continuous modules is that complements are summands. Modules with this special property are called extending modules or C1 modules. We investigate decomposition properties of dual-extending (D1) modules, those modules which are supplemented and for which each supplement is a summand. The notions of hollowness and dual Goldie dimension play a prominent role. Our results are analogous to results for extending modules developed by Camillo and Yousif.

Generalizing results known for continuous and quasicontinuous modules, Camillo and Yousif showed in [1] that if $M$ is a C1 module whose socle has finite Goldie dimension then $M$ may be decomposed as the direct sum of a finite dimensional module and a semisimple module. Much of their work depends on the properties of essentially closed modules. In section 1 we develop a dual notion, the concept of the interior of a submodule. This in turn leads to the definition of open submodules. In section 3 we show that the concept of dual Goldie dimension developed by Grzeszczuk and Puczylowski allows us to formulate a decomposition criterion involving the Jacobson radical of the module.

All rings are assumed to have unity. Unless stated otherwise, all modules are assumed to be unitary right $R$-modules. We follow the convention that the intersection of the empty family of submodules of a module $M$ is $M$ itself.

The symbol $<$ denotes proper inclusion; where equality is permitted we write $\leq$. The set of all elements belonging to set $A$ but not to $B$ is denoted $A \setminus B$. If $a \in A$ we write $A - a$ rather than $A - \{a\}$.

1. Supplements and Interiors

In this section we use the concept of smallness, which is dual to essentiality, to develop a concept dual to essential closure. We begin with some basic definitions. See [8].
Definition 1.1. A \( \leq M \) is small in \( M \), denoted \( A \xrightarrow{o} M \), provided that, for each \( B \leq M \), \( A + B = M \) implies that \( B = M \). \( M \) is hollow if every proper submodule of \( M \) is small.

Definition 1.2. Let \( A \leq M \). \( B \) is a supplement of \( A \) in \( M \) provided that \( B \) is minimal among submodules \( C \) of \( M \) such that \( A + C = M \).

Lemma 1.3. Suppose that \( B + B^* = M \). Then \( B^* \) is a supplement of \( B \) in \( M \) if and only if \( B \cap B^* \xrightarrow{o} B^* \).

Proof. \( \Rightarrow \): Suppose that \( B \cap B^* + C = B^* \). Then \( M = B + B^* = B + B \cap B^* + C = B + C \). Since \( C \leq B^* \), \( B^* \) is a supplement of \( B \) in \( M \), we have \( C = B^* \).

\( \Leftarrow \): Suppose that \( C \leq B^* \) and \( B + C = M \). Then \( B^* \leq B \cap B^* + C \leq B^* \).

Recall that \( C \leq M \) is a complement of \( A \leq M \) if \( C \) is maximal among submodules \( B \leq M \) such that \( A \cap B = 0 \). A Zorn's lemma argument shows that every submodule of a given module has a complement in that module.

Definition 1.4. A module \( M \) is supplemented if for any two submodules \( A \) and \( B \) with \( A + B = M \), \( B \) contains a supplement of \( A \) in \( M \).

The notion of supplement is dual to that of complement. Note that not every submodule of \( M \) need have a supplement. Hence, not all modules are supplemented.

Definition 1.5. A module \( M \) is semiprimitive provided \( \text{Rad } M = 0 \). If every factor of \( M \) is semiprimitive, we say that \( M \) is completely semiprimitive.

Proposition 1.6. Let \( M \) be a supplemented module. If \( M \) is semiprimitive then \( M \) is semisimple.

Proof. Let \( B \leq M \). Since \( M \) is supplemented, there exists \( B^* \leq M \) such that \( B + B^* = M \) and \( B \cap B^* \xrightarrow{o} B^* \). Since \( M \) is semiprimitive, it has no nonzero small submodule. Hence \( M = B \oplus B^* \).

Thus every submodule of \( M \) is a summand.

Since any semisimple module is semiprimitive, the proposition above shows that for supplemented modules the notions of semiprimitivity and semisimplicity are equivalent.

The following proposition, whose proof is straightforward, gathers some relevant facts about supplemented modules.

Proposition 1.7. Any factor of a supplemented module is supplemented. Any summand of a supplemented module is supplemented. If \( M \) is a supplemented module then \( \frac{M}{\text{Rad } M} \) is semisimple.

If \( A \leq X \leq M \) then the notation \( A \xrightarrow{\star} X \) denotes that \( A \) is an essential submodule of \( X \), and we say that \( X \) is an essential extension of \( A \) in \( M \).